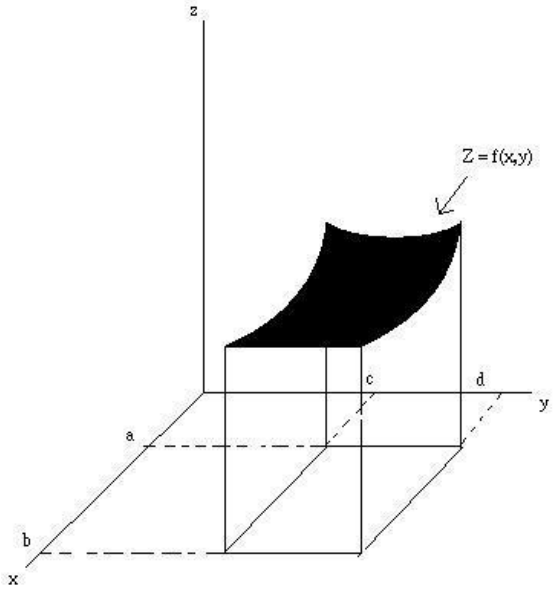
Multiple Integral

Double Integrals and Volumes

Let $R = [a, b] \times [c, d], S = \{(x, y, z) \in R^3 \mid (x, y) \in R, 0 \le z \le f(x, y)\}$

To find the volume of S - V(S)



Similarly to define $\int_a^b f(x)dx$

Divide the rectangle R into subintervals

$[a, b]$ is divided into m subinterval $[x_{i-1}, x_i]$

$i = 1,2,\cdots m$, $\Delta x_i = x_i - x_{i-1}$, and $[c, d]$ is divided into n subinterval $[y_{j-1}, y_j]$, $\Delta y_j = y_j - y_{j-1}$, $j = 1,2,\cdots n$

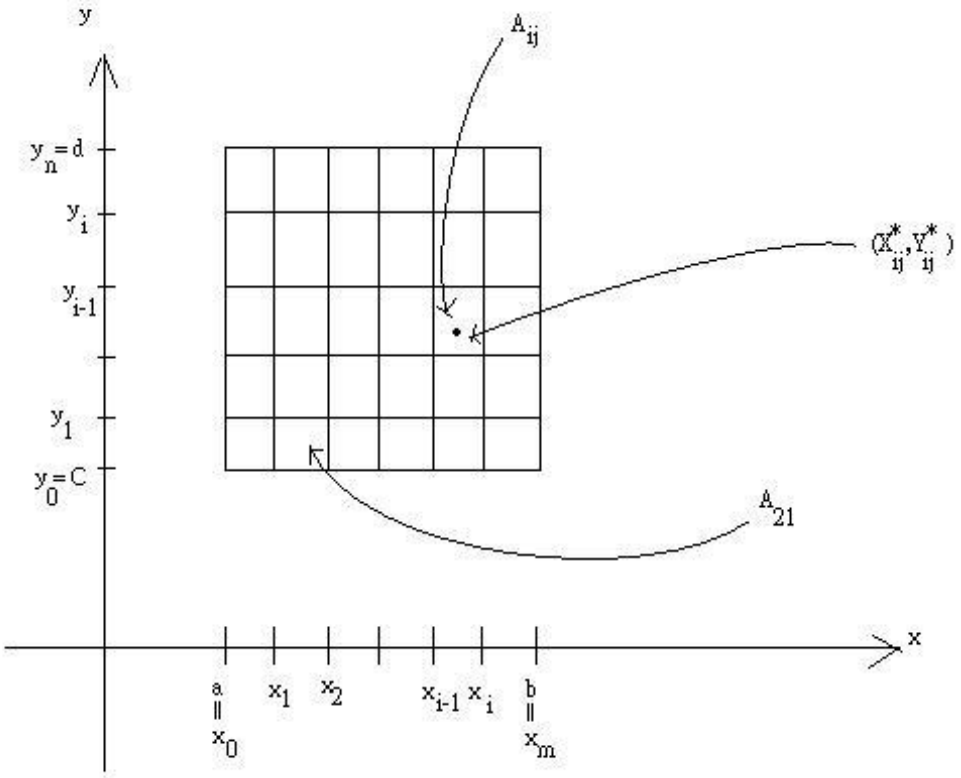
Define $R_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j]$ $i = 1,\cdots m$; $j = 1,\cdots n$

The area of R_{ij} is $\Delta A_{ij} = \Delta x_i \Delta y_j$

choose a sample point (x_{ij}^*, y_{ij}^*) in each R_{ij}

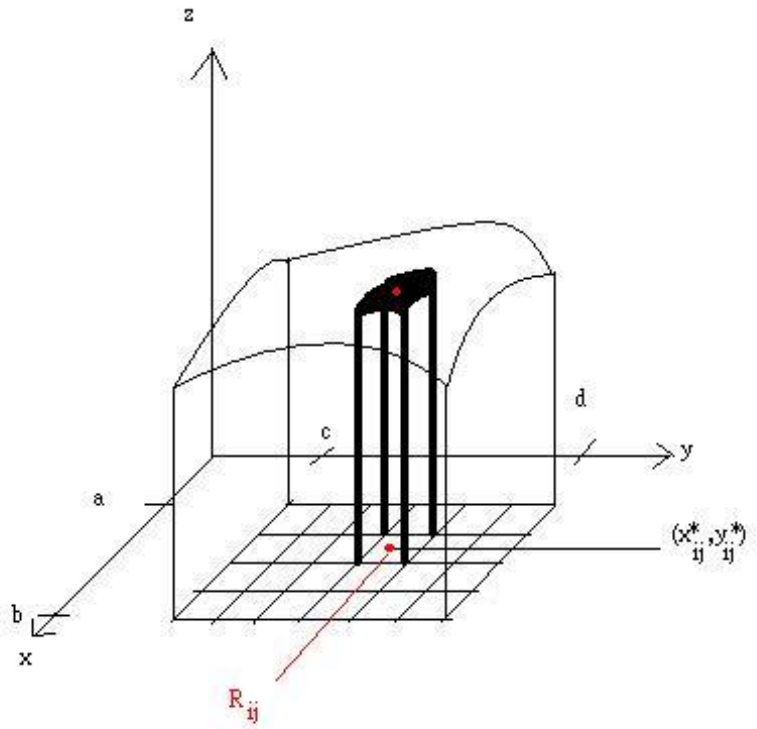
The volume of S can approximate $\sum_{j=1}^{n} \sum_{i=1}^{m} f(x_{ij}^*, y_{ij}^*) \Delta A_{ij}$

i.e $V(S) \approx \sum_{j=1}^{n} \sum_{i=1}^{m} f(x_{ij}^*, y_{ij}^*) \Delta A_{ij}$



Let $|P|$ denote the length of the longest diagonal

$R_{ij}, i = 1,2\cdots m;\ j = 1,2\cdots n$



Definition :

The double integral of f over the rectangle R is $\iint_R f(x, y)dA$

$$\iint_R f(x, y)dA = \lim_{|P|\to 0} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x^*_{ij}, y^*_{ij})\Delta A_{ij}$$

if this limit exists

properties:

1. $\iint_R cf(x, y)dA = c\iint_R f(x, y)dA$

2. $\iint_R (f(x, y) + g(x, y))dA = \iint_R f(x, y)dA + \iint_R g(x, y)dA$

3. If $f(x, y) \geq g(x, y) \forall (x, y) \in R$, then

$$\iint_R f(x, y)dA \geq \iint_R g(x, y)dA$$

Definition :

1. f is integrable on R, if $\lim_{|P|\to 0} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^{*}, y_{ij}^{*}) \Delta A_{ij}$ exist

2. $\iint_{R} f(x, y) dA$ is called the double integral of f over R

Theorem 1

Let f be bounded on the closed rectabgle R

(i) If f is continuous on R, then f is integrable on R

(ii) If f is continuous on R except on a finite number of smooth curves, then f is integrable on R

Definition :

1. If $\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$, then f is continuous at (a,b)
2. If f is continuous at all $(a,b) \in R$, then f is continuous on R

Example :

1. $f(x,y) = \sin xy, (x,y) \in R = [0,\pi] \times [0,2\pi]$

   f is continuous on R

2. $f(x,y) = x^2 y + x,\ R = [0,\infty) \times [0,\infty)$

   f is continuous on R

3. $$f(x,y) = \begin{cases} \dfrac{y}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

   f is not continuous on $(0,1)$

   f is not continuous on $(o,w), \forall w$

Iterated Integrals

$$\text{Fixed } y, \text{Let } A(y) = \int_1^3 x^2 y \, dx = y \cdot \frac{x^3}{3}\Big|_1^3 = \frac{26}{3} y$$

$$\text{Consider } \int_0^2 A(y) dy = \int_0^2 \left(\int_1^3 x^2 y \, dx \right) dy$$

Remark :

$$\text{For function } f(x, y), R = [a, b] \times [c, d]$$

1. $$\text{Let } A(x) = \int_c^d f(x, y) dy, B(y) = \int_a^b f(x, y) dx$$

$$\int_a^b A(x) dx = \int_a^b \left(\int_c^d f(x, y) dy \right) dx \text{ is called an iterated integral}$$

2. $\int_a^b \int_c^d f(x, y)\,dy\,dx = \int_a^b \left(\int_c^d f(x, y)\,dy\right)dx$

3. $\int_c^d \int_a^b f(x, y)\,dy\,dx = \int_c^d \left(\int_a^b f(x, y)\,dy\right)dx$

Example : evaluate

(i) $\int_0^3 \int_1^4 x^2 y\,dx\,dy$ (ii) $\int_0^3 \int_1^4 x^2 y\,dy\,dx$

(iii) $\int_0^4 \int_0^8 \frac{1}{4}(64 - 8x + y^2)\,dy\,dx$

(iv) $\int_0^8 \int_0^4 \frac{1}{4}(64 - 8x + y^2)\,dx\,dy$

(v) $\int_0^3 \int_1^2 (3x + 2y)\,dx\,dy$

Theorem 2 (Fubini's Theorem)

If f is continuous on $R = [a, b] \times [c, d]$, the

$$\iint_R f(x, y)dA = \int_c^d \int_a^b f(x, y)dxdy = \int_a^b \int_c^d f(x, y)dy\,dx$$

Ex :

1. Find $\iint_R (x - 3y^2)dA$, where $R = [0,2] \times [1,3]$

2. Find $\iint_R y\sin xy\,dA$, where $R = [0,2] \times [0, \pi]$

Ans : $\iint_R y\sin xy\,dA = \int_0^2 \int_0^\pi y\sin xy\,dy\,dx = ?$

But $\int_0^\pi \int_0^2 y\sin xy\,dxdy = \int_0^\pi (-\cos xy)\Big|_0^2 dy = \int_0^\pi (1 - \cos 2y)dy$

$= \pi - (\frac{1}{2}\sin 2y\Big|_0^\pi) = \pi - 0 = \pi$

3. Find $\iint_R x\cos xy\,dA$, where $R = [0, \pi] \times [0, \pi]$

4. Find $\iint_R \frac{1+y}{1+x}dA$, where $R = \{(x, y) | -1 \le x \le 2, 0 \le y \le 1\}$

Exercises

1. Let $R = \{(x, y) | 1 \le x \le 3, 0 \le y \le 2\}$, Evaluate $\iint_R f(x, y)dA$, where f is the given function

1. $$f(x, y) = \begin{cases} 2, 1 \le x \le 3, 0 \le y \le 2 \\ 3, 3 < x \le 4, 0 \le y \le 2 \end{cases}$$

2. $$f(x, y) = \begin{cases} 2, 1 \le x \le 4, 0 \le y \le 1 \\ 1, 1 \le x < 3, 1 \le y \le 2 \\ 4, 3 \le x \le 4, 1 \le y \le 2 \end{cases}$$

2. If $R = [0,1] \times [0,1]$, show that

$$D \le \iint_R \sin(x + y)dA \le 1$$

3. Let $R_1 = \{(x, y) | 0 \le x \le 2, 0 \le y \le 1\}, R = [0,2] \times [0,2]$

$R_2 = \{(x, y) | 0 \le x \le 2, 1 \le y \le 2\}$

Suppose that $\iint_R f(x, y)dA = 4, \iint_R g(x, y)dA = 6, \iint_{R_2} g(x, y)dA = 2$

Evaluate

1. $\iint_R (4f(x, y) - g(x, y))dA$
2. $\iint_{R_1} 3g(x, y)dA$
3. $\iint_R (2f(x, y) + 4)dA$
4. $\iint_R (3f(x, y) + 4g(x, y))dA$

4. Evaluate each of the interated integrals

(i) $\int_0^2 \int_1^3 f(x, y)\,dy\,dx$, where $f(x, y) = x^2 y$

(ii) $\int_0^1 \int_0^1 xe^{xy}\,dx dy$

(iii) $\int_0^{\pi} \int_0^1 x \sin y\,dx dy$

(iv) $\int_0^2 \int_0^2 \frac{y}{1+x^2}\,dy\,dx$

(v) $\int_{-2}^2 \int_{-1}^1 |x^2 y^3|\,dy\,dx$

(vi) $\int_{-2}^2 \int_{-1}^1 [x^2] y^3\,dy\,dx$

(vii) $\int_{-2}^2 \int_{-1}^1 [x]\,|y|\,dy\,dx$

5. Calculate the double integral

(i) $\iint_R (2y^2 - 3xy^3)dA, R = \{(x, y) | 1 \le x \le 2, 0 \le y \le 3\}$

(ii) $\iint_R x\cos(x + y)dA, R = [0, \frac{\pi}{6}] \times [0, \frac{\pi}{3}]$

(iii) $\iint_R \frac{1+x}{2+y} dA, R = \{(x, y) | -1 \le x \le 2, 0 \le y \le 1\}$

(iv) $\iint_R xy e^{x^2 y^2} dA, R = [0,1] \times [0,1]$

(v) $\iint_R xy\sqrt{1+x^2} dA, R = \{(x, y) | 0 \le x \le \sqrt{3}, 1 \le y \le 2\}$

6. Find the volume of the solid lying under the plane

$Z = 2x + 5y + 1$ and above the rectangle

$\{(x, y) | (-1 \le x \le 0, 1 \le y \le 4)$

7. Evaluate the interated integral

(i) $\int_0^1 \int_0^x \cos x^2 \, dy \, dx$

(ii) $\int_0^1 \int_{1-x}^{2+x} (3x - y^2) dy \, dx$

(iii) $\int_0^1 \int_{3y}^3 e^{x^2} dx dy$

(iv) $\int_0^1 \int_{\sqrt{y}}^1 \sqrt{1 + x^3} dx dy$

(v) $\int_0^1 \int_0^{y^2} 2y e^x dx dy$

(vi) $\int_0^4 \int_{x/2}^3 e^{y^2} dy \, dx$

8. Evaluate $\iint_D \sqrt{1 - x^2 - y^2}\, dA$, where $D = \{(x, y) | x^2 + y^2 \leq 1\}$

9. Find the volume of the given solid

(i) Bounded by the cylinders $x^2 + y^2 = 2^2$ and

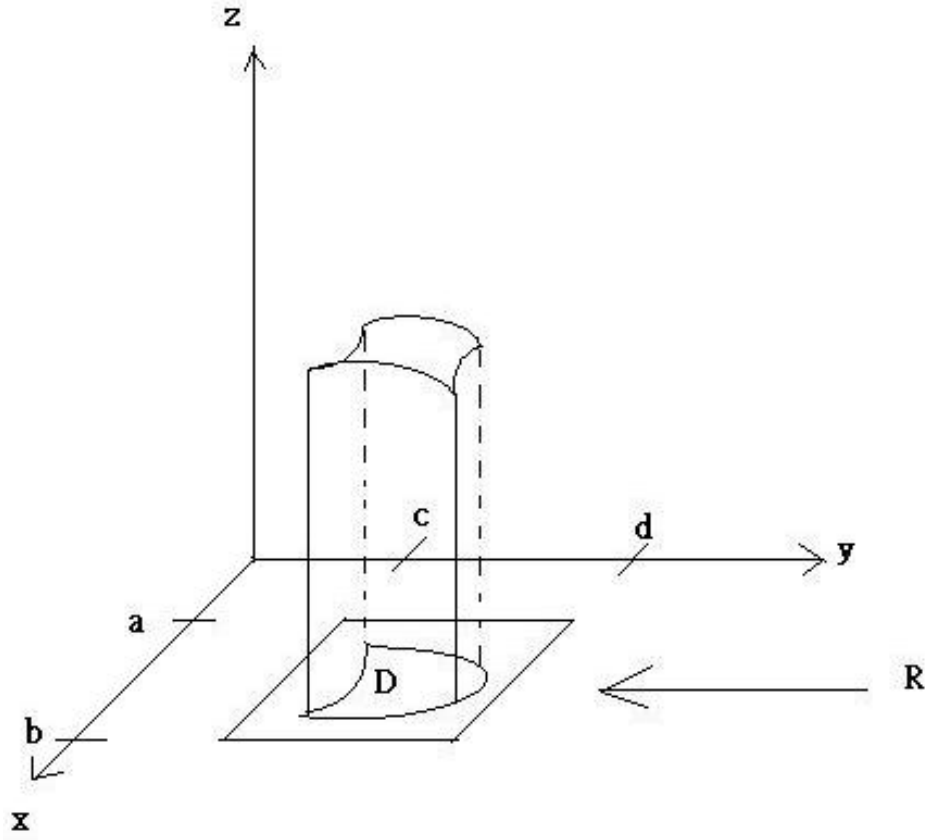
$y^2 + z^2 = 2^2$

(ii) Bounded by the planes $x = 0, y = 0, z = 0$

and $x + y + z = 1$

Double Integral over General Regions

Let D be a bounded region and $D \subset R$, f is a function defined on D. Define a new function



$$F(x, y) = \begin{cases} f(x, y) & \text{if } (x, y) \in D \\ 0 \text{ if } (x, y) \text{is in R but not in D} \end{cases}$$

Definition :

1. The double integral of f over D is

$$\iint_D f(x, y)dA = \iint_R F(x, y)dA$$

2. A plane region D is said to be of type I if

$$D = \{(x, y) | a \le x \le b, g_1(x) \le y \le g_2(x)\},$$

where g_1, g_2 are two continuous function.

3. A plane region D is said to be of type II if

$$D = \{(x, y) | h_1(y) \le x \le h_2(y), c \le y \le d\},$$

where h_1, h_2 are two continuous function.

Example :

1. $D_1 = \{(x, y) | 0 \le x \le \pi, \sin x \le y \le 1\}$, TypeI

2. $D_2 = \{(x, y) | -1 \le y \le 1, 2y^2 \le x \le 1 + y^2\}$, TypeII

Properties:

1. If f is continuous on a typeI region D such that

$D = \{(x, y) | a \le x \le b, g_1(x) \le y \le g_2(x)\}$,

then $\iint_D f(x, y)dA = \int_a^b \int_{g_{1(x)}}^{g_2(x)} f(x, y)dy\,dx$

2. If f is continuous on a typeII region D then

$\iint_D f(x, y)dA = \int_c^d \int_{h_{1(y)}}^{h_2(y)} f(x, y)dx\,dy$

where $D = \{(x, y) | h_1(y) \le x \le h_2(y), c \le y \le d\}$

Example :

1. Evaluate $\iint_D (x+3y)dA$

Where $D=\{(x,y)\,|\,-1\le x\le 1,\ 2x^2\le y\le 1+x^2\}$

Ans :

$$\iint_D (x+3y)dA=\int_{-1}^{1}\int_{2x^2}^{1+x^2}(x+3y)dy\,dx$$

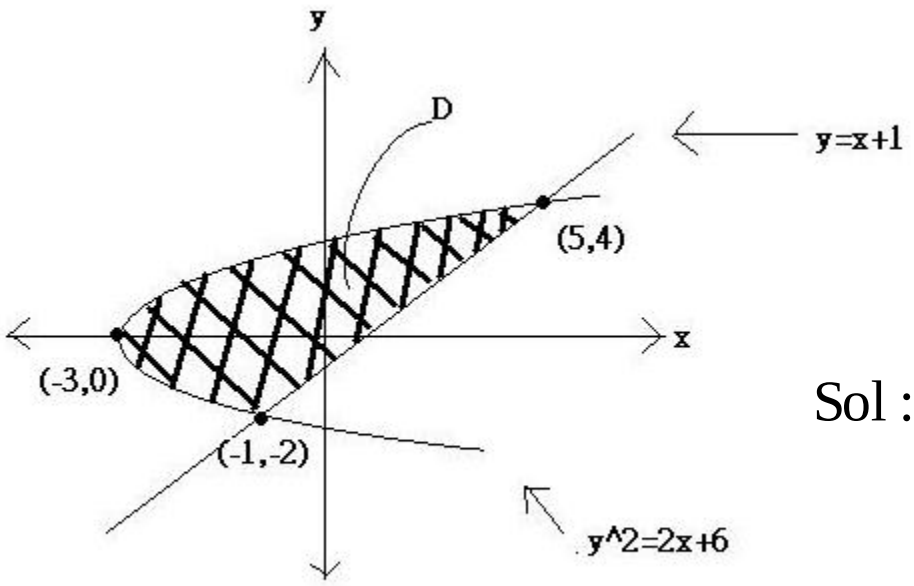
$$=\int_{-1}^{1} x(1+x^2-2x^2)+\frac{3}{2}((1+x^2)^2-(2x^2)^2)dx$$

$$=\int_{-1}^{1} x+x^3-2x^3+\frac{3}{2}+3x^2+\frac{3}{2}x^4-4x^4dx$$

$$=\left(\frac{1}{2}x^2-\frac{1}{4}x^4+\frac{3}{2}x+x^3-\frac{1}{2}x^5\right)\Bigg|_{-1}^{1}=\frac{3}{2}+1-\frac{1}{2}=2$$

2. Evaluate $\iint_D xy\,dA$ where D is the region bounded by

the line $y = x - 1$ and the parabola $y^2 = 2x + 6$



Sol :

$$D = \{(x, y) \mid -3 \le x \le 5, ? \le y \le \sqrt{2x+6}\}$$

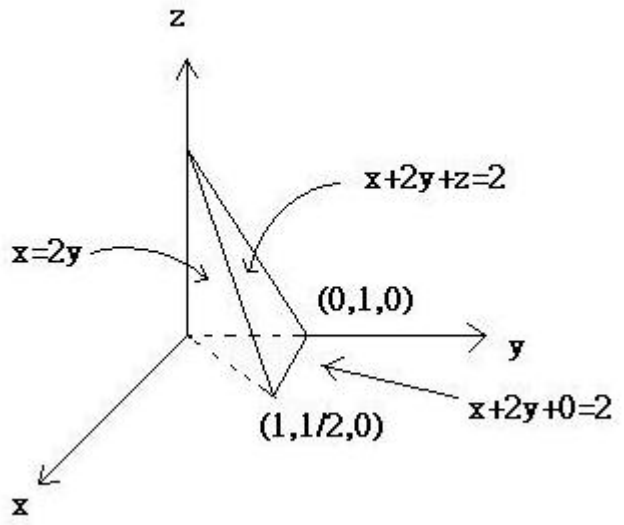
$$= \{(x, y) \mid \frac{y^2 - 6}{2} \le x \le y + 1, -2 \le y \le 4\}$$

$$\iint_D xy\,dA = \int_{-2}^{4} \int_{\frac{y^2-6}{2}}^{y+1} xy\,dx\,dy = 36$$

3. Find the volume of the tetrahedron bounded by the planes

$x = 2y, x = 0, z = 0$ and $x + 2y + z = 2$

Sol :



$$D = \{(x, y) \mid 0 \le x \le 1, \frac{x}{2} \le y \le \frac{2-x}{2}\}$$

$$\text{所求 } V = \iint_D 2 - x - 2y \, dA = \int_0^1 \int_{\frac{x}{2}}^{\frac{2-x}{2}} (2 - x - 2y) \, dy \, dx$$

$$= \frac{1}{3}$$

4. Evaluate $\int_0^1 \int_x^1 \sin(y^2)\,dy\,dx$; $\frac{1}{2}(1-\cos 1)$

5. Evaluate $\int_0^1 \int_x^1 \cos(y^2)\,dy\,dx$

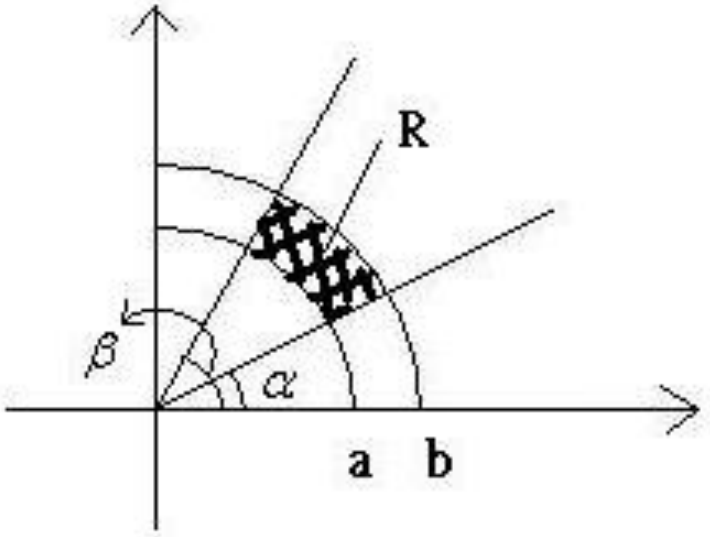
Sol :

$$D = \{(x, y) \mid 0 \le x \le 1,\ x \le y \le 1\}$$

$$= \{(x, y) \mid 0 \le y \le 1,\ 0 \le x \le y\}$$

Double Integrals in Polar Coordinates

Consider $\mathrm{R} = \{(\mathrm{r}, \theta) \mid \mathrm{a} \le \mathrm{r} \le \mathrm{b}, \alpha \le \theta \le \beta$



Polar rectangle

Example：

$$1.\ \mathrm{R} = \{(\mathrm{r},\theta) \mid 0 \le \mathrm{r} \le 1,\ 0 \le \theta \le 2\pi\}$$

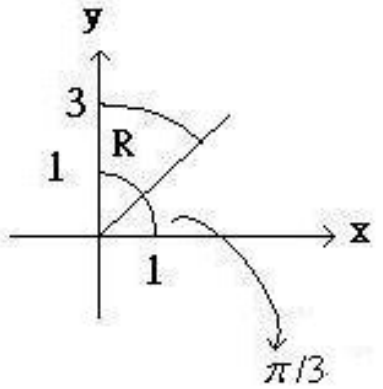
$$2.\ \mathrm{R} = \{(\mathrm{r},\theta) \mid 1 \le \mathrm{r} \le 3,\ 0 \le \theta \le \pi\}$$

$$3.\ \mathrm{R} = \{(\mathrm{r},\theta) \mid 1 \le \mathrm{r} \le 3,\ \frac{\pi}{3} \le \theta \le \frac{\pi}{2}\}$$

$$\text{The area of R is } \mathrm{A(R)} = (\pi \cdot 3^2 - \pi \cdot 1^2)\frac{\frac{\pi}{2} - \frac{\pi}{3}}{2\pi}$$

$$= \frac{1}{2}(3^2 - 1^2) \cdot (\frac{\pi}{2} - \frac{\pi}{3})$$

$$= \frac{2}{3}\pi$$



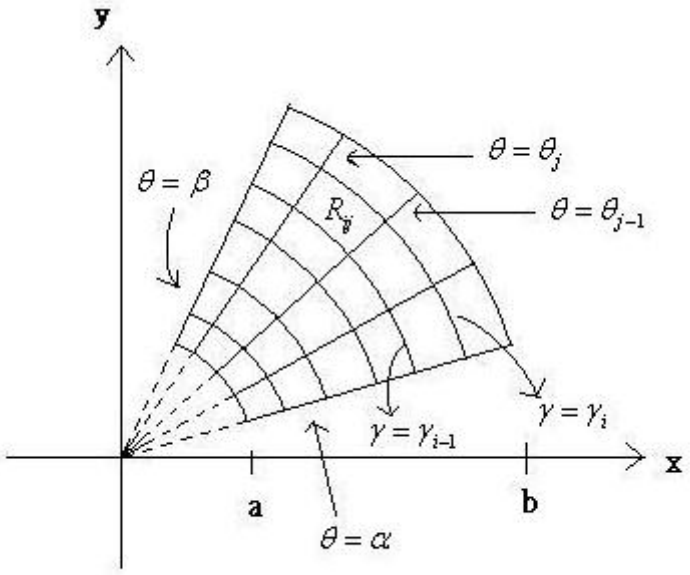
4. $R_{ij} = \{(r,\theta) \mid r_{i-1} \le r \le r_i, \theta_{j-1} \le \theta \le \theta_j\}$

The area of R_{ij} - ΔA_{ij} is

$$\Delta A_{ij} = \frac{1}{2} r_i^2 \Delta\theta_j - \frac{1}{2} r_{i-1}^2 \Delta\theta_j = \frac{1}{2}(r_i + r_{i-1})(r_i - r_{i-1})\Delta\theta_j$$

$$= r_i^* \Delta r_i \Delta\theta_j$$

Where $\Delta r_i = r_i - r_{i-1}, \Delta\theta_j = \theta_j - \theta_{j-1}, i = 1, \cdots m;\ j = 1, \cdots n$



The Riemanu sum of f on R is

$$\sum_{i=1}^{m}\sum_{j=1}^{n} f(r_i^* \cos\theta_j^*, r_i^* \sin\theta_j^*)\Delta A_{ij}$$

$$= \sum_{i=1}^{m}\sum_{j=1}^{n} f(r_i^* \cos\theta_j^*, r_i^* \sin\theta_j^*) r_i^* \Delta r_i \Delta\theta_j$$

$$\to \int_{\alpha}^{\beta}\int_{a}^{b} f(r\cos\theta, r\sin\theta) r\,dr\,d\theta$$

Properties

1. Let $R = \{(r,\theta) \mid a \le r \le b, \alpha \le \theta \le \beta\}$ be a polar rectangle and $0 \le \beta - \alpha \le 2\pi$ If f is continuous on R, then

$$\iint_R f(x, y)dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)rdrd\theta$$

2. Let $D = \{(r,\theta) \mid \alpha \le \theta \le \beta, h_1(\theta) \le r \le h_2(\theta)\}$ be a polor region. If f is continuous on D then

$$\iint_D f(x, y)dA = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta, r\sin\theta)rdrd\theta$$

Example :

1. Evaluate $\iint_R (4y^2 + 3x)dA$

where $R = \{(x, y) | \, y \geq 0, 1 \leq x^2 + y^2 \leq 4\}$

Sol :

$$R = \{(x, y) | \, y \geq 0, 1 \leq x^2 + y^2 \leq 4\}$$

$$= \{(r, \theta) | \, 1 \leq r \leq 2, 0 \leq \theta \leq \pi\}$$

$$\iint_R (4y^2 + 3x)dA = \int_0^{\pi} \int_1^2 (4(r\sin\theta)^2 + 3r\cos\theta)rdrd\theta$$

$$= \int_0^{\pi} (15\sin^2\theta + 7\cos\theta)d\theta$$

$$= \frac{15}{2}\pi$$

2. Find the volume of the solid bounded by the plane $z = 0$ and the paraboloid $z = 1 - x^2 - y^2$

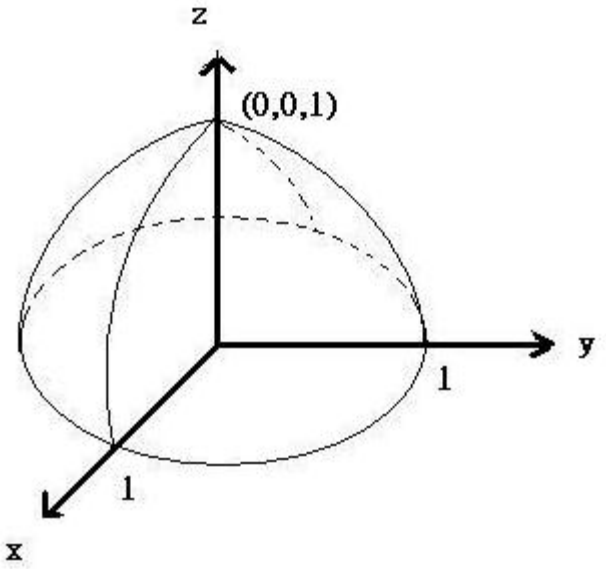
Sol :

$$D = \{(r, \theta) \mid 0 \le r \le 1, 0 \le \theta \le 2\pi\}$$

$$V = \iint_D (1 - x^2 - y^2)\,dA$$

$$= \int_0^{2\pi} \int_0^1 (1 - r^2)\,r\,dr\,d\theta$$

$$= \frac{\pi}{2}$$



Example :

$$\text{Evaluate} \iint_{R^2} e^{-(x^2+y^2)} dA$$

$$\text{where } R^2 = \{(x, y) \mid -\infty < x < \infty, -\infty < y < \infty\}$$

Sol :

$$\text{Consider } D_n = \{(r, \theta) \mid 0 \le r \le n, 0 \le \theta \le 2\pi\}$$

$$\iint_{R^2} e^{-(x^2+y^2)} dA = \lim_{n\to\infty} \iint_{D_n} e^{-(x^2+y^2)} dA$$

$$= \lim_{n\to\infty} \int_0^{2\pi} \int_0^n e^{-r^2} r\, dr\, d\theta = \lim_{n\to\infty} \int_0^{2\pi} \left(\frac{1}{2} - \frac{1}{2} e^{-n^2}\right) d\theta$$

$$= \pi$$