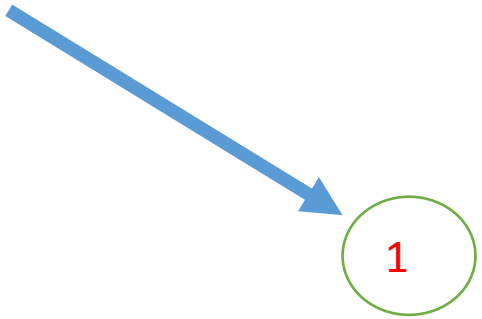
Newton's backward interpolation formula

Given the set of $(n+1)$ values.
(x_0, y_0), (x_1, y_1),……. (x_n, y_n) of x and y.
It is required to find $y(x)$ a polynomial of n^{th} degree such that $y(x)$ agree at the tabulated values.

$$\text{Let } x_i = x_0 + ih, \quad i = 0,1,2,3, \ldots \ldots . n.$$

Assume that the function $y(x)$ is a polynomial of n^{th} degree and given by

$$y(x) = A_0 + A_1(x - x_n) + A_2(x - x_n)\,(x - x_{n-1}) + A_3(x - x_n)\,(x - x_{n-1})$$
$$(x - x_{n-2}) + \ldots\ldots\ldots + A_n(x - x_n)\,(x - x_{n-1}) \ldots\ldots (x - x_2)\,(x - x_1)$$



Where $A_0, A_1, A_2, \ldots\ldots\ldots A_n$ are constants to be determine.

$$y(x) = A_0 + A_1(x - x_n) + A_2(x - x_n)(x - x_{n-1}) + A_3(x - x_n)(x - x_{n-1})$$
$$(x - x_{n-2}) + \ldots\ldots\ldots + A_n(x - x_n)(x - x_{n-1}) \ldots\ldots (x - x_2)(x - x_1)$$

At $x = x_n, y(x_n) = y_n = A_0$

$\Rightarrow A_0 = y_n$

At $x = x_{n-1}$,

$$y(x_{n-1}) = A_0 + A_1(x_{n-1} - x_n)$$

i.e $y_{n-1} = y_n - A_1(x_{n-1} - x_n)$

$$\therefore A_1 = \frac{y_n - y_{n-1}}{h} = \frac{\nabla y_n}{h}$$

$$y(x) = A_0 + A_1(x - x_n) + A_2(x - x_n)\,(x - x_{n-1}) + A_3(x - x_n)\,(x - x_{n-1})\,(x - x_{n-2}) + \ldots\ldots\ldots + A_n(x - x_n)\,(x - x_{n-1}) \ldots\ldots (x - x_2)\,(x - x_1)$$

At point $x = x_{n-2}$,

$$y(x_{n-2}) = A_0 + A_1(x_{n-2} - x_n) + A_2\,(x_{n-2} - x_n)(x_{n-2} - x_{n-1})$$

$$y_{n-2} = y_n + \frac{\nabla y_n}{h}(x_{n-2} - x_n) + A_2\,(x_{n-2} - x_n)(x_{n-2} - x_{n-1})$$

$$\therefore y_{n-2} - y_n = \frac{y_n - y_{n-1}}{h}(-2h) + 2h^2 A_2$$

$$\therefore \nabla^2 y_n = 2h^2 A_2$$

$$A_2 = \frac{y_{n-2} - 2y_{n-1} + y_n}{2h^2} = \frac{\nabla^2 y_n}{2h^2}$$

Continuing in this way we get, $A_n = \frac{\nabla^n y_n}{n! h^n}$

$$x_i = x_0 + ih$$

Setting, $x = x_n + ph$

$$(x - x_n) = ph$$

$$(x - x_{n-1}) = x - (x_n - h) = x - x_n + h = ph + h = (p+1)h$$

$$(x - x_{n-2}) = x - (x_{n-1} - h) = x - x_{n-1} + h = (p+1)h + h = (p+2)h$$

$$(x - x_1) = [p + (n-1)]h$$

$$y(x) = A_0 + A_1(x - x_n) + A_2(x - x_n)(x - x_{n-1}) + A_3(x - x_n)(x - x_{n-1})$$
$$(x - x_{n-2}) + \ldots\ldots\ldots + A_n(x - x_n)(x - x_{n-1}) \ldots \ldots (x - x_2)(x - x_1)$$

Substituting $A_0, A_1, \ldots A_n$ in equation (1), we get

$$y(x) = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n + \cdots + \frac{p(p+1)(p+2)\ldots(p+n-1)}{n!}\nabla^n y_n$$

This is Newton's backward interpolation formula.

Ex.1) From the following data, find the value of $\tan 17^0$

θ	0	4	8	12	16	20	24
$\tan\theta$	0	0.0699	0.1405	0.2126	0.2867	0.3640	0.4402

First we form the backward difference table

θ	$\tan\theta$	∇	∇^2	∇^3	∇^4	∇^5	∇^6
0	0						
		0.699					
4	8		0.007				
		0.0706		-0.00055			
8	0.1405		0.00015		0.0024		
		0.0721		0.00185		-0.00305	
12	0.2126		0.002		-0.00065		-0.0018
		0.0741		0.0012		-0.00485	
16	0.2867		0.0032		-0.0055		
		0.0773		-0.0043			
20	0.3640		-0.0011				
		0.0762					
24	0.4402						

We have by Newton's backward interpolation formula

$$y(x)$$

$$= y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n + \frac{p(p+1)(p+2)(p+3)}{4!}\nabla^4 y_n$$

$$+ + \frac{p(p+1)(p+2)(p+3)(p+4)}{5!}\nabla^5 y_n + \frac{p(p+1)(p+2)(p+3)(p+4)(p+5)}{6!}\nabla^6 y_n$$

Here $x_n = 24, h = 4, x = 17$

$$p = \frac{x - x_n}{h} = \frac{17 - 24}{4} = -1.75$$

$$y(17) = 0.4402 + (-1.75)(0.0762) + \frac{(-1.75)(-0.75)}{2}(-0.0011)$$

$$+ \frac{(-1.75)(-0.75)(0.25)}{6}(-0.0043) + \frac{(-1.75)(-0.75)(0.25)(1.25)}{24}(-0.0055)$$

$$+ \frac{(-1.75)(-0.75)(0.25)(1.25)(2.25)}{120}(-0.00485)$$

$$+ \frac{(-1.75)(-0.75)(0.25)(1.25)(2.25)}{720}($$

θ	$\tan\theta$	∇	∇^2	∇^3	∇^4	∇^5	∇^6
0	0						
		0.699					
4	8		0.007				
		0.0706		-0.00055			
8	0.1405		0.00015		0.0024		
		0.0721		0.00185		-0.00305	
12	0.2126		0.002		-0.00065		-0.0018
		0.0741		0.0012		-0.00485	
16	0.2867		0.0032		-0.0055		
		0.0773		-0.0043			
20	0.3640		-0.0011				
		0.0762					
24	0.4402						

$$= 0.4402 - 0.13335 - 0.00072 - 0.00023 - 0.00009 - 0.000037 - 0.000007$$

$$= 0.30576$$

Ex.2) The population of a town in census is given below. Estimate the population for the year 1955.

Year	1921	1931	1941	1951	1961
Population (in thousand)	46	66	81	93	101

First we form the backward difference table

Year	Population (in thousand)	∇	∇^2	∇^3	∇^4
1921	46				
		20			
1931	66		-5		
		15		2	
1941	81		-3		-3
		12		-1	
1951	93		-4		
		8			
1961	101				

We have by Newton's backward interpolation formula

$$y(x) = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n + \frac{p(p+1)(p+2)(p+3)}{4!}\nabla^4 y_n$$

Here $x_n = 1961, h = 10, x = 1955$

$$p = \frac{x - x_n}{h} = \frac{1955 - 1961}{10} = -0.6$$

$$y(1955)$$

$$= 101 + (-0.6)8 + \frac{(-0.6)(0.4)}{2}(-4) + \frac{(-0.6)(0.4)(1.4)}{6}(-1)$$

$$+ \frac{(-0.6)(0.4)(1.4)(2.4)}{24}(-3)$$

$$= 101 - 4.8 + 0.48 + 0.056 + 0.1008$$

$$= 95.5632$$

Year	Population (in thousand)	∇	∇^2	∇^3	∇^4
1921	46				
		20			
1931	66		-5		
		15		2	
1941	81		-3		-3
		12		-1	
1951	93		-4		
		8			
1961	101				

Ex.3) The number of students who obtained marks in an examination are given in the following table:

Marks	0-19	20-39	49-59	60-79	80-99
No. of students	41	62	65	50	17

Find number of students who obtained less than 70 marks.

First we form the backward difference table

Less than marks	No. of students	∇	∇^2	∇^3	∇^4
19	41				
		62			
39	103		3		
		65		-18	
59	168		-15		0
		50		-18	
79	218		-33		
		17			
99	235				

We have by Newton's backward interpolation formula

$$y(x) = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n + \frac{p(p+1)(p+2)(p+3)}{4!}\nabla^4 y_n$$

Here $x_n = 99, h = 20, x = 70$

$$p = \frac{x - x_n}{h} = \frac{70 - 99}{20} = -1.45$$

$$y(1955) = 235 + (-1.45)17 + \frac{(-1.45)(-0.45)}{2}(-33) + \frac{(-1.45)(-0.45)(0.55)}{6}(-18)$$

$$= 235 - 24.65 - 10.7662 - 1.0766 + 0$$

$$= 198.5072 \approx 198$$

Less than marks	No. of students	∇	∇^2	∇^3	∇^4
19	41				
		62			
39	103		3		
		65		-18	
59	168		-15		0
		50		-18	
79	218		-33		
		17			
99	235				

Ex.4) From the following table, find y when $x = 2.4$

x	1.7	1.8	1.9	2.0	2.1	2.2	2.3
e^x	5.474	6.050	6.686	7.389	8.166	9.025	9.974

First we form the backward difference table

x	e^x	∇	∇^2	∇^3	∇^4	∇^5	∇^6
1.7	5.474						
		0.576					
1.8	6.050		0.06				
		0.636		0.007			
1.9	6.686		0.067		0		
		0.703		0.007		0.001	
2.0	7.389		0.074		0.001		-0.002
		0.777		0.008		-0.001	
2.1	8.166		0.082		0		
		0.859		0.008			
2.2	9.025		0.09				
		0.949					
2.3	9.974						

We have by Newton's backward interpolation formula

Here $x_n = 2.3, h = 0.1, x = 2.4$

$y(x)$

$$= y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n + \frac{p(p+1)(p+2)(p+3)}{4!}\nabla^4 y_n +$$

$$\frac{p(p+1)(p+2)(p+3)(p+4)}{5!}\nabla^5 y_n + \frac{p(p+1)(p+2)(p+3)(p+4)(p+5)}{6!}\nabla^6 y_n$$

$$p = \frac{x - x_n}{h} = \frac{2.4 - 2.3}{0.1} = 1$$

$$= 9.974 + 0.949 + 0.09 + 0.008 + 0 - 0.001 - 0.002$$

$$= 11.018$$

Ex.5) From the following table, find $y(7.5)$

x	1	2	3	4	5	6	7	8
y	1	8	27	64	125	216	343	512

First we form the backward difference table

x	y	∇	∇^2	∇^3
1	1			
		7		
2	8		12	
		19		6
3	27		18	
		37		6
4	64		24	
		61		6
5	125		30	
		91		6
6	216		36	
		127		6
7	343		42	
		169		
8	512			

We have by Newton's backward interpolation formula

$$y(x) = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n$$

Here $x_n = 8, h = 1, x = 7.5$

$$p = \frac{x - x_n}{h} = \frac{7.5 - 8}{1} = -0.5$$

$$= 512 - 84.5 - 5.25 - 0.375$$

$$= 421.875$$