Newton's Interpolation formulae

These interpolation formulae are based on taking finite forward differences of the given tabulated functional values at equally spaced arguments at small interval h

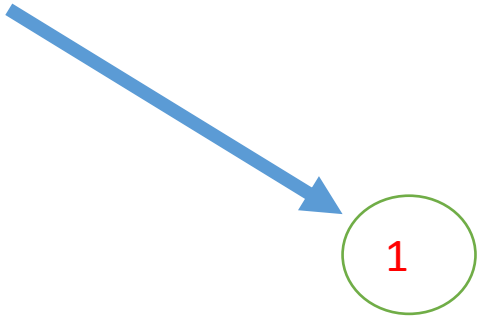
Newton's forward interpolation formula

Given the set of $(n+1)$ values.
(x_0, y_0), (x_1, y_1),……. (x_n, y_n) of x and y.
It is required to find $y(x)$ a polynomial of n^{th} degree such that $y(x)$ agree at the tabulated values.

$$\text{Let } x_i = x_0 + ih, \quad i = 0,1,2,3, \ldots \ldots . n.$$

Assume that the function $y(x)$ is a polynomial of n^{th} degree and given by

$$y(x) = A_0 + A_1(x - x_0) + A_2(x - x_0)(x - x_1) + A_3(x - x_0)(x - x_1)$$
$$(x - x_2) + \ldots\ldots\ldots + A_n(x - x_0)(x - x_1)(x - x_2) + \ldots\ldots\ldots + (x - x_{n-1})$$



Where $A_0, A_1, A_2, \ldots\ldots\ldots A_n$ are constants to be determine.

$$y(x) = A_0 + A_1(x - x_0) + A_2(x - x_0)\ (x - x_1) + A_3(x - x_0)\ (x - x_1)$$
$$(x - x_2) + \ldots\ldots\ldots + A_n(x - x_0)\ (x - x_1)(x - x_2) + \ldots\ldots\ldots + (x - x_{n-1})$$

At $x = x_0, y(x_0) = y_0 = A_0$

$\Rightarrow A_0 = y_0$

At $x = x_1$,

$$y(x_1) = A_0 + A_1(x_1 - x_0)$$

i.e $y_1 = y_0 + A_1(x_1 - x_0)$

$$\therefore A_1 = \frac{y_1 - y_0}{x_1 - x_0} = \frac{\Delta y_0}{h}$$

$$y(x) = A_0 + A_1(x - x_0) + A_2(x - x_0)\ (x - x_1) + A_3(x - x_0)\ (x - x_1)$$
$$(x - x_2) + \ldots\ldots\ldots + A_n(x - x_0)\ (x - x_1)(x - x_2) + \ldots\ldots\ldots + (x - x_{n-1})$$

At point $x = x_2$,

$$y(x_2) = A_0 + A_1(x_2 - x_0) + A_2\ (x_2 - x_0)(x_2 - x_1)$$

$$y_2 = y_0 + \frac{y_1 - y_0}{x_1 - x_0}(x_2 - x_0) + A_2\ (x_2 - x_0)(x_2 - x_1)$$

$$\therefore y_2 - y_0 = \frac{y_1 - y_0}{h}(2h) + 2h^2 A_2$$

$$\therefore y_2 - y_0 - 2y_1 + 2y_0 = 2h^2 A_2$$

$$A_2 = \frac{y_2 - 2y_1 + y_0}{2h^2} = \frac{\Delta^2 y_0}{2h^2}$$

Continuing in this way we get, $A_n = \frac{\Delta^n y_0}{n! h^n}$

$$x_i = x_0 + ih$$

Setting, $x = x_0 + ph$

$$(x - x_0) = ph$$

$$(x - x_1) = x - (x_0 + h) = x - x_0 - h = ph - h = (p - 1)h$$

$$(x - x_2) = x - (x_0 + 2h) = x - x_0 - 2h = ph - 2h = (p - 2)h$$

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$$(x - x_{n-1}) = [p - (n - 1)]h$$

$$y(x) = A_0 + A_1(x - x_0) + A_2(x - x_0)\ (x - x_1) + A_3(x - x_0)\ (x - x_1)$$
$$(x - x_2) + \ldots\ldots\ldots + A_n(x - x_0)\ (x - x_1)(x - x_2) + \ldots\ldots\ldots + (x - x_{n-1})$$

Substituting $A_0, A_1, \ldots A_n$ in equation (1), we get

$$y(x) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \cdots + \frac{p(p-1)(p-2)\ldots(p-(n-1)}{n!}\Delta^n y_0$$

This is Newton's forward interpolation formula.

Ex.1) Find the cubic polynomial which takes the following values :

x	0	1	2	3
$y = f(x)$	1	2	1	10

First we will prepare the forward difference table .

x	y	Δ	Δ^2	Δ^3
0	1			
		1		
1	2		-2	
		-1		12
2	1		10	
		9		
3	10			

x	y	Δ	Δ^2	Δ^3
0	1			
		1		
1	2		-2	
		-1		12
2	1		10	
		9		
3	10			

Here $x_0 = 0, h = 1$

$$p = \frac{x - x_0}{h} = \frac{x - 0}{1} = x$$

By Newton's interpolation formula,

$$y(x) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0$$

$$y(x) = 1 + x(1) + \frac{x(x-1)}{2}(-2) + \frac{x(x-1)(x-2)}{6}(12)$$

$$y(x) = 2x^3 - 7x^2 + 6x + 1$$

Ex.2) If l_x represents the number of persons living at age x in a life table, find accurately l_x for $x = 35$.

Given $l_{20} = 512, l_{30} = 390, l_{40} = 360, l_{50} = 243$

First we will prepare the forward difference table .

x	l_x	Δl_x	$\Delta^2 l_x$	$\Delta^3 l_x$
20	512			
		-122		
30	390		152	
		30		-65
40	360		87	
		117		
50	243			

x	l_x	Δl_x	$\Delta^2 l_x$	$\Delta^3 l_x$
20	512			
		-122		
30	390		152	
		30		-65
40	360		87	
		117		
50	243			

Here $x_0 = 20, h = 10, x = 35$

$$p = \frac{x - x_0}{h} = \frac{35 - 20}{10} = 1.5$$

By Newton's interpolation formula,

$$y(x) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0$$

$$y(35) = 512 + 1.5(-122) + \frac{1.5(0.5)}{2}(152) + \frac{(1.5)(0.5)(-0.5)}{6}(-65)$$

$$y(35) = 390.06 \approx 390$$

Ex.3) From the following data, find the number of students who obtained less than 45 marks.

Marks	30 - 40	40 - 50	50 - 60	60 - 70	70 - 80
No. of students	31	42	51	35	31

We construct the forward difference table as follows:

Marks	No. of students	Δ	Δ^2	Δ^3	Δ^4
Below 40	31				
		42			
Below 50	31+42=73		9		
		51		-25	
Below 60	51+73=124		-16		37
		35		12	
Below 70	35+124=159		-4		
		31			
Below 80	31+159=190				

Marks	No. of students	Δ	Δ^2	Δ^3	Δ^4
Below 40	31				
		42			
Below 50	31+42 =73		9		
		51		-25	
Below 60	51+73 =124		-16		37
		35		12	
Below 70	35+124 =159		-4		
		31			
Below 80	31+159 =190				

Here $x_0 = 40, h = 10, x = 45$

$$p = \frac{x - x_0}{h} = \frac{45 - 40}{10} = 0.5$$

By Newton's forward interpolation formula,

$$y(x) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \frac{p(p-1)(p-2)(p-3)}{4!}\Delta^4 y_0$$

$$y(45) = 31 + 0.5\frac{0.5(-0.5)}{2}(9) + \frac{(0.5)(-0.5)(-1.5)}{6}(-25) + \frac{(0.5)(-0.5)(-1.5)(-2.5)}{24}(37)$$

$$y(45) = 47.8671 \approx 48$$

Ex.4) The following are the data from steam table:

Temperature ^0C	140	150	160	170	180
Pressure kgf/cm^2	3.685	4.854	6.302	8.076	10.225

Find pressure for a temperature of 142^0C.

We construct the forward difference table as follows:

Temperature ^0C	Pressure kgf/cm^2	Δ	Δ^2	Δ^3	Δ^4
140	3.685				
		1.169			
150	4.854		0.279		
		1.448		0.047	
160	6.302		0.326		0.002
		1.774		0.049	
170	8.076		0.375		
		2.149			
180	10.225				

Here $x_0 = 140, h = 10, x = 142$

$$p = \frac{x - x_0}{h} = \frac{142 - 140}{10} = 0.2$$

Temperature ^0C	Pressure kgf/cm^2	Δ	Δ^2	Δ^3	Δ^4
140	3.685				
		1.169			
150	4.854		0.279		
		1.448		0.047	
160	6.302		0.326		0.002
		1.774		0.049	
170	8.076		0.375		
		2.149			
180	10.225				

By Newton's forward interpolation formula,

$$y(x) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \frac{p(p-1)(p-2)(p-3)}{4!}\Delta^4 y_0$$

$$y(142) = 3.685 + 0.2(1.169) + \frac{0.2(-0.8)}{2}(0.279) + \frac{(0.2)(-0.8)(-1.8)}{6}(0.047) + \frac{(0.2)(-0.8)(-1.8)(-2.8)}{24}(0.002)$$

$$= 3.8987$$

Temperature ^0C	Pressure kgf/cm^2	Δ	Δ^2	Δ^3	Δ^4
140	3.685				
		1.169			
150	4.854		0.279		
		1.448		0.047	
160	6.302		0.326		0.002
		1.774		0.049	
170	8.076		0.375		
		2.149			
180	10.225				

Ex.5) The population of a town in the decimal census was given below. Estimate the population for the year 1895

Year	1891	1901	1911	1921	1931
Population	46	66	81	93	101

We construct the forward difference table as follows:

Year	Population	Δ	Δ^2	Δ^3	Δ^4
1891	46				
		20			
1901	66		-5		
		15		2	
1911	81		-3		-3
		12		-1	
1921	93		-4		
		8			
1931	101				

Here $x_0 = 1891, h = 10, x = 1895$

$$p = \frac{x - x_0}{h} = \frac{1895 - 1891}{10} = 0.4$$

Year	Population	Δ	Δ^2	Δ^3	Δ^4
1891	46				
		20			
1901	66		-5		
		15		2	
1911	81		-3		-3
		12		-1	
1921	93		-4		
		8			
1931	101				

By Newton's forward interpolation formula,

$$y(x) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \frac{p(p-1)(p-2)(p-3)}{4!}\Delta^4 y_0$$

$$y(1895) = 46 + 0.4(20) + \frac{0.4(-0.6)}{2}(-5) + \frac{(0.4)(-0.6)(-1.6)}{6}(2) + \frac{(0.4)(-0.6)(-1.6)(-2.6)}{24}(-3)$$

$= 54.8528$

Year	Population	Δ	Δ^2	Δ^3	Δ^4
1891	46				
		20			
1901	66		-5		
		15		2	
1911	81		-3		-3
		12		-1	
1921	93		-4		
		8			
1931	101				