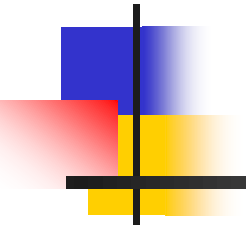
Chapter 4 Linear Transformations



4.1 Introduction to Linear Transformations

4.2 The Kernel and Range of a Linear Transformation

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4.4 Transition Matrices and Similarity

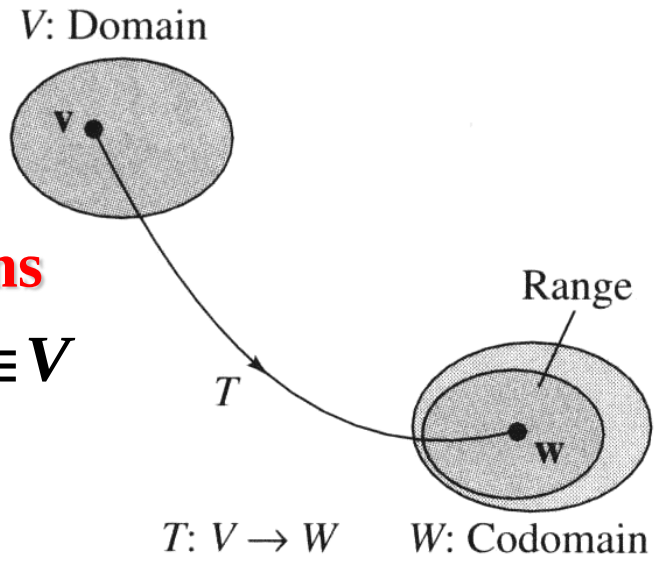
4.1 Introduction to Linear Transformations

- A linear transformation is a function T that maps a vector space V into another vector space W:

$$T: V \xrightarrow{\text{mapping}} W, \qquad V, W: \text{vector space}$$

V: the domain of T

W: the co-domain of T



Two axioms of linear transformations

(1) $T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v}), \quad \forall \mathbf{u}, \mathbf{v} \in V$

(2) $T(c\mathbf{u}) = cT(\mathbf{u}), \quad \forall c \in R$

- Image of **v** under T:

  If **v** is in V and **w** is in W such that

  $$T(\mathbf{v}) = \mathbf{w}$$

  Then **w** is called the image of **v** under T .

- **the range of T:**

  The set of all images of vectors in V.

  $$range(T) = \{T(\mathbf{v}) \mid \forall \mathbf{v} \in V\}$$

- the pre-image of **w**:

  The set of all **v** in V such that $T(\mathbf{v})=\mathbf{w}$.

- Notes:

(1) A **linear transformation** is said to be **operation preserving**.

$$T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v}) \qquad T(c\mathbf{u}) = cT(\mathbf{u})$$

Addition in V | Addition in W | Scalar multiplication in V | Scalar multiplication in W

(2) A linear transformation $T: V \to V$ from **a vector space into itself** is called a **linear operator**.

- Ex: Verifying a linear transformation T from R^2 into R^2

$$T(v_1, v_2) = (v_1 - v_2, v_1 + 2v_2)$$

Pf:

$\mathbf{u} = (u_1, u_2)$, $\mathbf{v} = (v_1, v_2)$: vector in R^2, c : any real number

(1) Vector addition :

$$\mathbf{u} + \mathbf{v} = (u_1, u_2) + (v_1, v_2) = (u_1 + v_1, u_2 + v_2)$$

$$\begin{aligned} T(\mathbf{u} + \mathbf{v}) &= T(u_1 + v_1, u_2 + v_2) \\ &= ((u_1 + v_1) - (u_2 + v_2), (u_1 + v_1) + 2(u_2 + v_2)) \\ &= ((u_1 - u_2) + (v_1 - v_2), (u_1 + 2u_2) + (v_1 + 2v_2)) \\ &= (u_1 - u_2, u_1 + 2u_2) + (v_1 - v_2, v_1 + 2v_2) \\ &= T(\mathbf{u}) + T(\mathbf{v}) \end{aligned}$$

(2) Scalar multiplication

$$c\mathbf{u} = c(u_1, u_2) = (cu_1, cu_2)$$

$$\begin{aligned} T(c\mathbf{u}) &= T(cu_1, cu_2) = (cu_1 - cu_2, cu_1 + 2cu_2) \\ &= c(u_1 - u_2, u_1 + 2u_2) \\ &= cT(\mathbf{u}) \end{aligned}$$

Therefore, T is a linear transformation.

- Ex: Functions that are not linear transformations

$(a)\ f(x) = \sin x$

$\sin(x_1 + x_2) \neq \sin(x_1) + \sin(x_2)$ $\Leftarrow$ **$f(x) = \sin x$ is not a linear transformation**

$\sin(\frac{\pi}{2} + \frac{\pi}{3}) \neq \sin(\frac{\pi}{2}) + \sin(\frac{\pi}{3})$

$(b)\ f(x) = x^2$

$(x_1 + x_2)^2 \neq x_1^2 + x_2^2$

$(1+2)^2 \neq 1^2 + 2^2$

$\Leftarrow$ **$f(x) = x^2$ is not a linear transformation**

$(c)\ f(x) = x + 1$

$f(x_1 + x_2) = x_1 + x_2 + 1$

$f(x_1) + f(x_2) = (x_1 + 1) + (x_2 + 1) = x_1 + x_2 + 2$

$f(x_1 + x_2) \neq f(x_1) + f(x_2)$ $\Leftarrow$ **$f(x) = x + 1$ is not a linear transformation**

- Notes: Two uses of the term “linear”.

(1) $f(x) = x + 1$ is called **a linear function** because its graph is a line. But

(2) $f(x) = x + 1$ is **not a linear transformation** from a vector space R into R **because it preserves neither vector addition nor scalar multiplication**.

- Zero transformation:

$$T: V \to W \qquad T(\mathbf{v}) = 0, \ \forall \mathbf{v} \in V$$

- Identity transformation:

$$T: V \to V \qquad T(\mathbf{v}) = \mathbf{v}, \ \forall \mathbf{v} \in V$$

- **Thm 4.1**: (Properties of linear transformations)

$$\boldsymbol{T: V \to W, \quad \mathbf{u}, \mathbf{v} \in V}$$

$$(1)\ T(\mathbf{0}) = \mathbf{0}$$

$$(2)\ T(-\mathbf{v}) = -T(\mathbf{v})$$

$$(3)\ T(\mathbf{u} - \mathbf{v}) = T(\mathbf{u}) - T(\mathbf{v})$$

(4) If $\mathbf{v} = c_1 v_1 + c_2 v_2 + \mathbf{L} + c_n v_n$

$$\text{Then } T(\mathbf{v}) = T(c_1 v_1 + c_2 v_2 + \mathbf{L} + c_n v_n)$$

$$= c_1 T(v_1) + c_2 T(v_2) + \mathbf{L} + c_n T(v_n)$$

- Ex: (Linear transformations and bases)

Let $T: R^3 \to R^3$ be a linear transformation such that

$T(1,0,0) = (2,-1,4)$

$T(0,1,0) = (1,5,-2)$

$T(0,0,1) = (0,3,1)$

Find $T(2, 3, -2)$.

Sol:

$$(2,3,-2) = 2(1,0,0) + 3(0,1,0) - 2(0,0,1)$$

$$\boldsymbol{T(2,3,-2) = 2T(1,0,0) + 3T(0,1,0) - 2T(0,0,1)} \quad (T \text{ is a L.T.})$$

$$\boldsymbol{= 2(2,-1,4) + 3(1,5,-2) - 2(0,3,1)}$$

$$\boldsymbol{= (7,7,0)}$$

- **Thm 4.2**: (The linear transformation given by a matrix)

  Let A be an $m \times n$ matrix. The function T defined by

  $$T(\mathbf{v}) = A\mathbf{v}$$

  is a linear transformation from R^n into R^m.

- Note:

R^n vector R^m vector

$$A\mathbf{v} = \begin{bmatrix} a_{11} & a_{12} & \mathbf{L} & a_{1n} \\ a_{21} & a_{22} & \mathbf{L} & a_{2n} \\ \mathbf{M} & \mathbf{M} & & \mathbf{M} \\ a_{m1} & a_{m2} & \mathbf{L} & a_{mn} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \mathbf{M} \\ v_n \end{bmatrix} = \begin{bmatrix} a_{11}v_1 + a_{12}v_2 + \mathbf{L} + a_{1n}v_n \\ a_{21}v_1 + a_{22}v_2 + \mathbf{L} + a_{2n}v_n \\ \mathbf{M} \\ a_{m1}v_1 + a_{m2}v_2 + \mathbf{L} + a_{mn}v_n \end{bmatrix}$$

$$T(\mathbf{v}) = A\mathbf{v}$$

$$T : R^n \longrightarrow R^m$$

- **A linear transformation from $M_{m\times n}$ into $M_{n\times m}$**

$$T(A)=A^T \quad (T: M_{m\times n}\rightarrow M_{n\times m})$$

Show that T is a linear transformation.

Sol:

$$A, B\in M_{m\times n}$$

$$T(A+B)=(A+B)^T=A^T+B^T=T(A)+T(B)$$

$$T(cA)=(cA)^T=cA^T=cT(A)$$

Therefore, T is a linear transformation from $M_{m\times n}$ into $M_{n\times m}$.

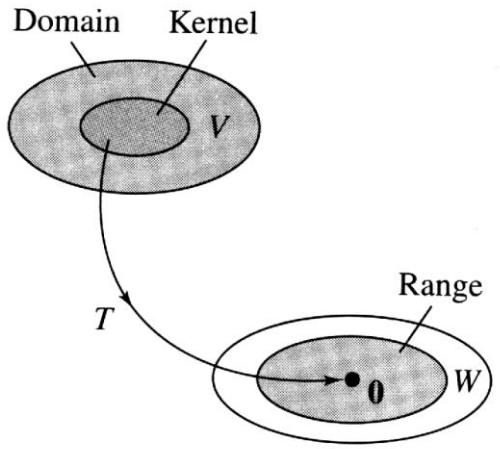
4.2 The Kernel and Range of a Linear Transformation

- **Kernel** of a linear transformation T:

  Let $T: V \to W$ be a linear transformation

  Then the set of all vectors $\mathbf{v}$ in V that satisfy $T(\mathbf{v}) = 0$ is called the **kernel** of T and is denoted by $\mathbf{ker}(T)$.

$$\mathbf{ker}(T) = \{\mathbf{v} \mid T(\mathbf{v}) = \mathbf{0},\ \forall \mathbf{v} \in V\}$$



Finding the kernel of a linear transformation

$$T(\mathbf{x}) = A\mathbf{x} = \begin{bmatrix} 1 & -1 & -2 \\ -1 & 2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \qquad (T : R^3 \to R^2) \qquad \ker(T) = ?$$

Sol:

$$\ker(T) = \{(x_1, x_2, x_3) \mid T(x_1, x_2, x_3) = (0,0),\ x = (x_1, x_2, x_3) \in R^3\}$$

$$T(x_1, x_2, x_3) = (0,0) \Rightarrow \begin{bmatrix} 1 & -1 & -2 \\ -1 & 2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & -2 & 0 \\ -1 & 2 & 3 & 0 \end{bmatrix} \xrightarrow{G.E} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} t \\ -t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \Rightarrow \ker(T) = \{t(1,-1,1) \mid t \in \text{real number}\}$$

$$= \text{span}\{(1,-1,1)\} = \text{Nullspace of A}$$

- **Thm 4.3: The kernel is a subspace of *V*.**

The kernel of a linear transformation $T: V \to W$ is a subspace of the domain *V*.

Pf: $\because T(\mathbf{0}) = \mathbf{0}$ **(Theorem 4.1)**

$\therefore \ker(T)$ is a nonempty subset of V

Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in the kernel of T. then

$$T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v}) = 0 + 0 = 0 \quad \Rightarrow \mathbf{u}+\mathbf{v} \in \ker(T)$$

$$T(c\mathbf{u}) = cT(\mathbf{u}) = c0 = 0 \quad \Rightarrow c\mathbf{u} \in \ker(T)$$

Thus, $\ker(T)$ is a subspace of V.

- **Corollary to Thm 4.3:**

Let $T: R^n \to R^m$ be the L.T given by $T(\mathbf{x}) = A\mathbf{x}$

Then the kernel of T is equal to the solution space of $A\mathbf{x} = 0$

$\boldsymbol{T(\mathbf{x}) = A\mathbf{x}}$ **(**a linear transformation $\boldsymbol{T: R^n \to R^m}$**)**

$\Rightarrow \boldsymbol{ker(T) = NS(A) = \{\mathbf{x} \mid A\mathbf{x} = \mathbf{0},\ \forall \mathbf{x} \in R^n\}}$ **(a subspace of $\boldsymbol{R^n}$)**

Finding a basis for the kernel

Let $T: R^5 \to R^4$ be defined by $T(\mathbf{x}) = A\mathbf{x}$, where $\mathbf{x}$ **is in** $\mathbf{R}^5$ and

$$A = \begin{bmatrix} 1 & 2 & 0 & 1 & -1 \\ 2 & 1 & 3 & 1 & 0 \\ -1 & 0 & -2 & 0 & 1 \\ 0 & 0 & 0 & 2 & 8 \end{bmatrix}$$

Find a basis for ker(T) as a subspace of R^5.

Sol:

$$[A \mid 0] = \begin{bmatrix} 1 & 2 & 0 & 1 & -1 & 0 \\ 2 & 1 & 3 & 1 & 0 & 0 \\ -1 & 0 & -2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 & 8 & 0 \end{bmatrix} \xrightarrow{G.E} \begin{bmatrix} 1 & 0 & 2 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 & -2 & 0 \\ 0 & 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(s and t mark the free columns)

$$\Rightarrow x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -2s + t \\ s + 2t \\ s \\ -4t \\ t \end{bmatrix} = s \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 2 \\ 0 \\ -4 \\ 1 \end{bmatrix}$$

$B = \{(-2, 1, 1, 0, 0)$ *and* $(1, 2, 0, -4, 1)\}$ is a basis of the kernel of T

- **Thm 4.4: The range of T is a subspace of W**

  The range of a linear transformation $T: V \to W$ is a subspace of W.

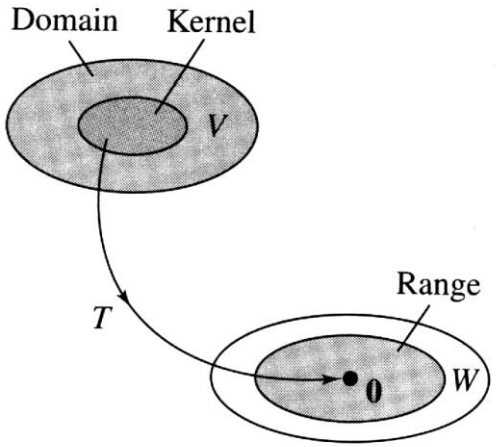
Pf: **Q $T(\mathbf{0}) = \mathbf{0}$ (Thm 4.1) $\Rightarrow$ *range*(T) is a nonempty subset of W**

Let $T(\mathbf{u})$ and $T(\mathbf{v})$ be vector in the range of T

$$T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v}) \in range(T) \qquad (\mathbf{u} \in V, \mathbf{v} \in V \Rightarrow \mathbf{u}+\mathbf{v} \in V)$$

$$T(c\mathbf{u}) = cT(\mathbf{u}) \in range(T) \qquad (\mathbf{u} \in V \Rightarrow c\mathbf{u} \in V)$$

Therefore, ***range*(T) is a subspace of W**



Let $T : R^n \to R^m$ be the L.T. represented by $T(\mathbf{x}) = A\mathbf{x}$,
then the range of T is equal to the column space of A

$\Rightarrow range(T) = CS(A) = \{Ax \mid \forall\ x \in R^n\}$

- Rank of a linear transformation T: $V \to W$:

  $rank(T) =$ **the dimension of the range of** T

- Nullity of a linear transformation T: $V \to W$:

  $nullity(T) =$ **the dimension of the kernel of** T

- **Note:**

  Let $T : R^n \to R^m$ be the L.T. represented by $T(\mathbf{x}) = A\mathbf{x}$, then

  $rank(T) = rank(A) = \dim[CS(A)]$

  $nullity(T) = nullity(A) = \dim[NS(A)]$

Finding a basis for the range of a linear transformation

Let $T: R^5 \to R^4$ be defined by $T(\mathbf{x}) = A\mathbf{x}$, where $\mathbf{x} \in R^5$ and

$$A = \begin{bmatrix} 1 & 2 & 0 & 1 & -1 \\ 2 & 1 & 3 & 1 & 0 \\ -1 & 0 & -2 & 0 & 1 \\ 0 & 0 & 0 & 2 & 8 \end{bmatrix}$$

Find a basis for the range($\boldsymbol{T}$).

Sol:

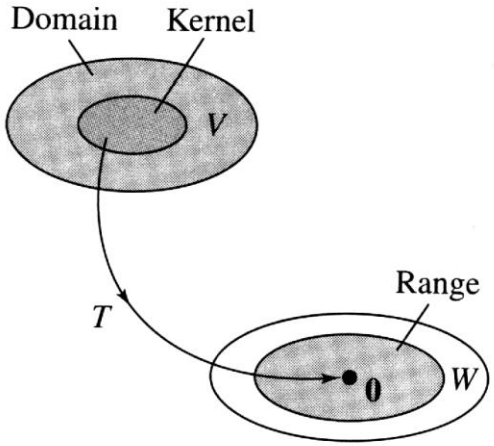
$$\boldsymbol{A} = \begin{bmatrix} \mathbf{1} & \mathbf{2} & \mathbf{0} & \mathbf{1} & \mathbf{-1} \\ \mathbf{2} & \mathbf{1} & \mathbf{3} & \mathbf{1} & \mathbf{0} \\ \mathbf{-1} & \mathbf{0} & \mathbf{-2} & \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{2} & \mathbf{8} \end{bmatrix} \xrightarrow{G.E} \begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{2} & \mathbf{0} & \mathbf{-1} \\ \mathbf{0} & \mathbf{1} & \mathbf{-1} & \mathbf{0} & \mathbf{2} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{4} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} = \boldsymbol{B}$$

$c_1 \quad c_2 \quad c_3 \quad c_4 \quad c_5$ $\qquad$ $w_1 \quad w_2 \quad w_3 \quad w_4 \quad w_5$

$\Rightarrow \{w_1, w_2, w_4\}$ is a basis for $CS(B)$

$\{c_1, c_2, c_4\}$ is a basis for $CS(A)$

$\Rightarrow \{(1, 2, -1, 0), (2, 1, 0, 0), (1, 1, 0, 2)\}$ is a basis for the range of T



- **Thm 4.5: Sum of rank and nullity**

Let $T: V \to W$ be a L.T. from an n-dimensional vector space V into a vector space W,

then $rank(T) + nullity(T) = n$,

***i.e.*, dim(range of T) + dim(kernel of T) = dim(domain of T)**

Pf: Let T **is represented by a *matrix* A_{mn}**

Assume $rank(A) = r$ (***i.e., the number of leading* var *iables***)

(1) $rank(T) = \dim(\text{range of } T) = \dim(\text{column space of } A)$

$= rank(A) = r$ Q $T(x) = Ax$

(2) $nullity(T) = \dim(\text{kernel of } T) = \dim(\text{solution space of } A)$

$= n - r$

$\Rightarrow rank(T) + nullity(T) = r + (n - r) = n$

- Finding the rank and nullity of a linear transformation

Find the rank and nullity of the L.T. $T : R^3 \to R^3$ define by

$$A = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Sol:

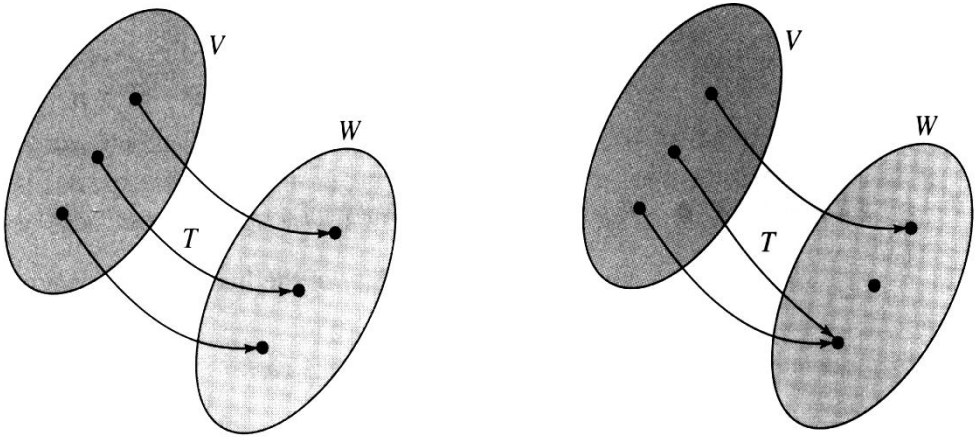
$$rank(T) = rank(A) = 2$$

$$nullity(T) = \dim(\text{domain of } T) - rank(T) = 3 - 2 = 1$$

- **One-to-one:**

A function $T: V \to W$ **is called one-to-one if the preimage of every w in the range of T consists of a single vector.**

T is one-to-one if and only if for all u and v in V, T(u)=T(v) implies that u=v.

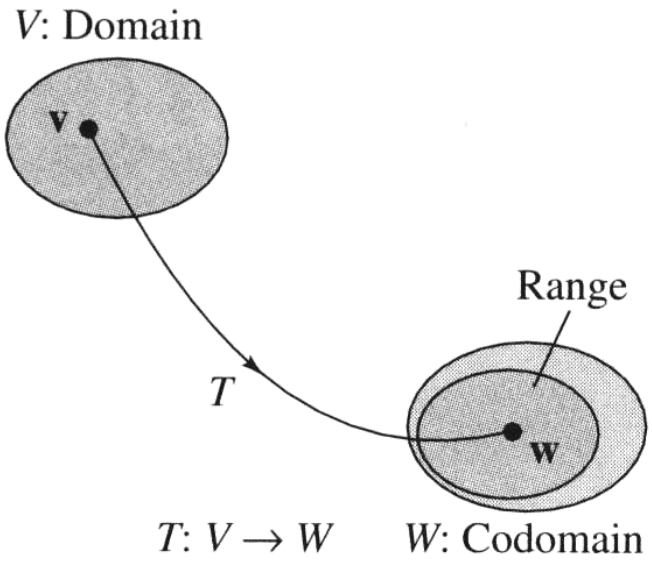


one-to-one not one-to-one

- **Onto:**

A function $T: V \to W$ is said to be onto if **every element in** W has a preimage in V

i.e., T **is onto** W **when** $range(T)=W$.



- **Thm 4.6:** (One-to-one linear transformation)

Let $T: V \to W$ be a L.T.,

T is one-to-one if and only if $\ker(T) = \{0\}$

i.e., The addtive unit element in V is mapped onto the additive unit element in W.

Pf: Suppose T is 1-1

Then $T(v) = 0$ can have only one solution : $v = 0$

i.e., $\ker(T) = \{0\}$

Suppose $\ker(T) = \{0\}$ and $T(u) = T(v)$

$T(u - v) = T(u) - T(v) = 0$

(T is a L.T.)

$\mathbf{Q}\ u - v \in \ker(T) \Rightarrow u - v = 0 \quad \Rightarrow T$ is one-to-one L.T.

- **One-to-one and not one-to-one linear transformation**

(*a*) The L.T. $T : M_{m\times n} \to M_{n\times m}$ **given by** $T(A) = A^T$

is one-to-one.

Because its kernel consists of only the $m \times n$ zero matrix

i.e., ker(T) = $\{0_{mn}\}$.

(*b*) The zero transformation $T : R^3 \to R^3$ is not one - to - one.

Because its kernel is all of R^3.

Note: $rank(T) = \dim[\text{range}(T)] = \dim[CS(A)]$

- **Onto linear transformation**

  Let $T: V \to W$ be a L.T., where W is finite dimensional,

  then T **is onto iff the rank of** T is equal to the dimension of W.

- **Thm 4.7**: (One-to-one and onto linear transformation)

  Let $T: V \to W$ be a L.T. with vector space V **and** W both of dimension n, then T is one-to-one iff it is onto.

Pf: If T is one-to-one, then $\ker(T) = \{0\}$ and $\dim(\ker(T)) = 0$

$rank(T) = \dim(range(T)) = n - \dim(\ker(T)) = n = \dim(W)$

Consequently, T is onto.

If T is onto, then $\dim(\text{range of } T) = \dim(W) = n$

$\dim(\ker(T)) = n - \dim(\text{range of } T) = n - n = 0$

Therefore, ker(T) = {0}. T is one-to-one. (from Thm 4.6)

Note: $rank(T)$ = the dimension of the range of T = $\dim(CS(A))$

- Ex:

The L.T. $\boldsymbol{T: R^n \to R^m}$ is given by $\boldsymbol{T(x) = Ax}$, find the nullity and rank of $\boldsymbol{T}$ and determine whether $\boldsymbol{T}$ is one-to-one, onto, or neither.

$$(a)A=\begin{bmatrix}1 & 2 & 0\\ 0 & 1 & 1\\ 0 & 0 & 1\end{bmatrix} \qquad (b)A=\begin{bmatrix}1 & 2\\ 0 & 1\\ 0 & 0\end{bmatrix}$$

$$(c)A=\begin{bmatrix}1 & 2 & 0\\ 0 & 1 & -1\end{bmatrix} \qquad (d)A=\begin{bmatrix}1 & 2 & 0\\ 0 & 1 & 1\\ 0 & 0 & 0\end{bmatrix}$$

Sol:

$T:R^n\to R^m$	dim(domain of T)	rank(T)	nullity(T)	1-1	onto
$(a)T:R^3\to R^3$	3	**3**	**0**	**Yes**	**Yes**
$(b)T:R^2\to R^3$	2	2	**0**	**Yes**	No
$(c)T:R^3\to R^2$	3	**2**	1	No	**Yes**
$(d)T:R^3\to R^3$	3	2	1	No	No

▪ Isomorphism

A linear transformation $T: V \to W$ that is one to one and onto is called an isomorphism. Moreover, if V **and** W are vector spaces such that there exists an isomorphism from V **to** W, then V **and** W are said to be isomorphic to each other.

▪ **Thm 4.8:** (Isomorphic spaces and dimension)

Two finite-dimensional vector space V and W are isomorphic if and only if they are of the same dimension.

Pf: Assume that V is isomorphic to W, where V has dimension n.

$\Rightarrow$ There exists a L.T. $T: V \to W$ that is one to one and onto.

Q T is one-to-one $\Rightarrow \dim(Ker(T)) = 0$

$$\Rightarrow \dim(\text{range of } T) = \dim(\text{domain of } T) - \dim(Ker(T))$$

$$= n - 0 = n$$

Q T **is onto.**

$\Rightarrow \dim(\text{range of } T) = \dim(W) = n$ Thus $\dim(V) = \dim(W) = n$

- Ex: (Isomorphic vector spaces)

The following vector spaces are isomorphic to each other.

$(a) R^4 = 4\text{-space}$

$(b) M_{4\times 1} = \text{space of all } 4\times 1 \text{ matrices}$

$(c) M_{2\times 2} = \text{space of all } 2\times 2 \text{ matrices}$

$(d) P_3(x) = \text{space of all polynomials of degree 3 or less}$

$(e) V = \{(x_1, x_2, x_3, x_4, 0), x_i \text{ is a real number}\} (\text{subspace of } R^5)$

4.3 Matrices for Linear Transformations

- **Two representations** of the linear transformation $T:R^3 \rightarrow R^3$:

$$(1) T(x_1, x_2, x_3) = (2x_1 + x_2 - x_3, -x_1 + 3x_2 - 2x_3, 3x_2 + 4x_3)$$

$$(2) T(\mathbf{x}) = A\mathbf{x} = \begin{bmatrix} 2 & 1 & -1 \\ -1 & 3 & -2 \\ 0 & 3 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

- Three reasons for **matrix representation** of a linear transformation:
  - It is simpler to write.
  - It is simpler to read.
  - It is more easily adapted for computer use.

- **Thm 4.9**: (**Standard matrix** for a linear transformation)

Let $T: R^n \to R^m$ be a linear transformation and $\{\mathbf{e}_1, \mathbf{e}_2, ..., \mathbf{e}_n\}$ are the basis of R^n such that

$$T(e_1) = \begin{bmatrix} a_{11} \\ a_{21} \\ \mathbf{M} \\ a_{m1} \end{bmatrix}, \quad T(e_2) = \begin{bmatrix} a_{12} \\ a_{22} \\ \mathbf{M} \\ a_{m2} \end{bmatrix}, \quad \mathbf{L}, \quad T(e_n) = \begin{bmatrix} a_{1n} \\ a_{2n} \\ \mathbf{M} \\ a_{mn} \end{bmatrix},$$

Then the $m \times n$ matrix whose i columns correspond to $T(e_i)$

$$A = \left[T(e_1) \mid T(e_2) \mid \mathbf{L} \mid T(e_n) \right] = \begin{bmatrix} a_{11} & a_{12} & \mathbf{L} & a_{1n} \\ a_{21} & a_{22} & \mathbf{L} & a_{2n} \\ \mathbf{M} & \mathbf{M} & \mathbf{O} & \mathbf{M} \\ a_{m1} & a_{m2} & \mathbf{L} & a_{mn} \end{bmatrix}$$

is such that $T(\mathbf{v}) = A\mathbf{v}$ for every $\mathbf{v}$ in R^n.

A is called the standard matrix for T.

Pf:

$$v \in R^n \Rightarrow v = \begin{bmatrix} v_1 \\ v_2 \\ \mathbf{M} \\ v_n \end{bmatrix} = v_1 \mathbf{r}_{e_1} + v_2 \mathbf{r}_{e_2} + \mathbf{L} + v_n \mathbf{r}_{e_n}$$

$$T \text{ is a L.T.} \Rightarrow T(\mathbf{v}) = T(v_1 \mathbf{r}_{e_1} + v_2 \mathbf{r}_{e_2} + \mathbf{L} + v_n \mathbf{r}_{e_n})$$

$$= T(v_1 \mathbf{r}_{e_1}) + T(v_2 \mathbf{r}_{e_2}) + \mathbf{L} + T(v_n \mathbf{r}_{e_n})$$

$$= v_1 T(\mathbf{r}_{e_1}) + v_2 T(\mathbf{r}_{e_2}) + \mathbf{L} + v_n T(\mathbf{r}_{e_n})$$

$$A\mathbf{v} = \begin{bmatrix} a_{11} & a_{12} & \mathbf{L} & a_{1n} \\ a_{21} & a_{22} & \mathbf{L} & a_{2n} \\ \mathbf{M} & \mathbf{M} & \mathbf{O} & \mathbf{M} \\ a_{m1} & a_{m2} & \mathbf{L} & a_{mn} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \mathbf{M} \\ v_n \end{bmatrix} = \begin{bmatrix} a_{11}v_1 + a_{12}v_2 + \mathbf{L} + a_{1n}v_n \\ a_{21}v_1 + a_{22}v_2 + \mathbf{L} + a_{2n}v_n \\ \mathbf{M} \\ a_{m1}v_1 + a_{m2}v_2 + \mathbf{L} + a_{mn}v_n \end{bmatrix}$$

$$= v_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \mathbf{M} \\ a_{m1} \end{bmatrix} + v_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \mathbf{M} \\ a_{m2} \end{bmatrix} + \mathbf{L} + v_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \mathbf{M} \\ a_{mn} \end{bmatrix}$$

$$= v_1 T(e_1) + v_2 T(e_2) + \mathbf{L} + v_n T(e_n)$$

Therefore, $T(\mathbf{v}) = A\mathbf{v}$ for each $\mathbf{v}$ in R^n

- Ex : (Finding the standard matrix of a linear transformation)

Find the standard matrix for the L.T. $\boldsymbol{T : R^3 \rightarrow R^2}$ define by

$$T(x, y, z) = (x - 2y, 2x + y)$$

Sol:

Vector Notation

$$T(e_1) = T(1, 0, 0) = (1, 2)$$

$$T(e_2) = T(0, 1, 0) = (-2, 1)$$

$$T(e_3) = T(0, 0, 1) = (0, 0)$$

Matrix Notation

$$T(e_1) = T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$T(e_2) = T\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$$T(e_3) = T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$A = \left[T(e_1) \mid T(e_2) \mid T(e_3) \right]$$

$$= \begin{bmatrix} 1 & -2 & 0 \\ 2 & 1 & 0 \end{bmatrix}$$

- Check:

$$A \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & -2 & 0 \\ 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x - 2y \\ 2x + y \end{bmatrix}$$

i.e., $\boldsymbol{T(x, y, z) = (x - 2y, 2x + y)}$

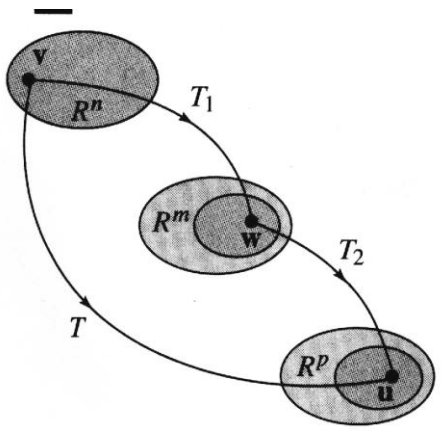
- Note:

$$A = \begin{bmatrix} 1 & -2 & 0 \\ 2 & 1 & 0 \end{bmatrix} \begin{array}{l} \leftarrow \; 1x - 2y + 0z \\ \leftarrow \; 2x + 1y + 0z \end{array}$$

- Composition of $T_1: R^n \to R^m$ with $T_2: R^m \to R^p$:

$$T(\mathbf{v}) = T_2(T_1(\mathbf{v})), \quad \mathbf{v} \in R^n$$

$$\boldsymbol{T = T_2 \circ T_1}, \quad \textbf{domain of } \boldsymbol{T} = \textbf{domain of } \boldsymbol{T_1}$$



Composition of Transformations

- **Thm 4.10:** (Composition of linear transformations)

Let $T_1 : R^n \to R^m$ and $T_2 : R^m \to R^p$ be L.T.
with standard matrices A_1 and A_2 , then

(1) The composition $\boldsymbol{T : R^n \to R^p}$, **defined by** $\boldsymbol{T(\mathbf{v}) = T_2(T_1(\mathbf{v}))}$, is a L.T.

(2) The standard matrix A for T is given by the matrix product $\boldsymbol{A = A_2 A_1}$

Pf:

(1)(T is a L.T.)

Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in R^n and let c be any scalar then

$$T(\mathbf{u}+\mathbf{v}) = T_2(T_1(\mathbf{u}+\mathbf{v})) = T_2(T_1(\mathbf{u})+T_1(\mathbf{v}))$$
$$= T_2(T_1(\mathbf{u}))+T_2(T_1(\mathbf{v})) = T(\mathbf{u})+T(\mathbf{v})$$

$$T(c\mathbf{v}) = T_2(T_1(c\mathbf{v})) = T_2(cT_1(\mathbf{v})) = cT_2(T_1(\mathbf{v})) = cT(\mathbf{v})$$

(2)(A_2A_1 is the standard matrix for T)

$$T(\mathbf{v}) = T_2(T_1(\mathbf{v})) = T_2(A_1\mathbf{v}) = A_2A_1\mathbf{v} = (A_2A_1)\mathbf{v}$$

- But note:

$$\boldsymbol{T_1 \circ T_2 \neq T_2 \circ T_1}$$

- Ex : (The standard matrix of a composition)

Let $\boldsymbol{T_1}$ and $\boldsymbol{T_2}$ be L.T. from $\boldsymbol{R^3}$ into $\boldsymbol{R^3}$ such that

$$T_1(x, y, z) = (2x + y, 0, x + z)$$

$$T_2(x, y, z) = (x - y, z, y)$$

Find the standard matrices for the compositions

$T = T_2 \circ T_1$ and $T' = T_1 \circ T_2$,

Sol:

$$A_1 = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} \quad (\text{standard matrix for } T_1)$$

$$A_2 = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad (\text{standard matrix for } T_2)$$

The standard matrix for $T = T_2 \text{ o } T_1$

$$A = A_2 A_1 = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

The standard matrix for $T' = T_1 \text{ o } T_2$

$$A' = A_1 A_2 = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & -2 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

- **Inverse linear transformation**

If $\boldsymbol{T_1 : R^n \to R^n}$ and $\boldsymbol{T_2 : R^n \to R^n}$ are L.T. such that for every $\mathbf{v}$ **in** $\boldsymbol{R^n}$

$$T_2(T_1(\mathbf{v})) = \mathbf{v} \quad \text{and} \quad T_1(T_2(\mathbf{v})) = \mathbf{v}$$

Then T_2 is called the inverse of T_1 and T_1 is said to be invertible

- Note:

If the transformation T is invertible, then the inverse is unique and denoted by T^{-1}.

- Existence of an inverse transformation

  Let $T : R^n \to R^n$ be a L.T. with standard matrix A,

  Then the following condition are equivalent.

  (1) T is invertible.

  (2) T is an isomorphism.

  (3) A is invertible.

- Note:

  If T is invertible with standard matrix A, then the standard matrix for T^{-1} is A^{-1}.

- Ex : (Finding the inverse of a linear transformation)

The L.T. $T: R^3 \to R^3$ is defined by

$$T(x_1, x_2, x_3) = (2x_1 + 3x_2 + x_3,\ 3x_1 + 3x_2 + x_3,\ 2x_1 + 4x_2 + x_3)$$

Show that T is invertible, and find its inverse.

Sol: The standard matrix for T

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 3 & 3 & 1 \\ 2 & 4 & 1 \end{bmatrix} \begin{matrix} \leftarrow 2x_1 + 3x_2 + x_3 \\ \leftarrow 3x_1 + 3x_2 + x_3 \\ \leftarrow 2x_1 + 4x_2 + x_3 \end{matrix}$$

$$\left[A \mid I_3\right] = \left[\begin{array}{ccc|ccc} 2 & 3 & 1 & 1 & 0 & 0 \\ 3 & 3 & 1 & 0 & 1 & 0 \\ 2 & 4 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{G.J.E} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 6 & -2 & -3 \end{array}\right] = \left[I \mid A^{-1}\right]$$

Therefore T is invertible and the standard matrix for T^{-1} is A^{-1}

$$A^{-1} = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 6 & -2 & -3 \end{bmatrix}$$

$$T^{-1}(\mathbf{v}) = A^{-1}\mathbf{v} = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 6 & -2 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_1 + x_2 \\ -x_1 + x_3 \\ 6x_1 - 2x_2 - 3x_3 \end{bmatrix}$$

In other words,

$$T^{-1}(x_1, x_2, x_3) = (-x_1 + x_2,\ -x_1 + x_3,\ 6x_1 - 2x_2 - 3x_3)$$

- **the matrix of T relative to the bases B and B'**

$$T : V \to W \qquad \text{(a L.T.)}$$

$$B = \{v_1, v_2, \mathrm{L}\ , v_n\} \qquad \text{(a basis for } V\text{)}$$

$$B' = \{w_1, w_2, \mathrm{L}\ , w_m\} \qquad \text{(a basis for } W\text{)}$$

Thus, the matrix of T relative to the bases B *and* B' is

$$A = \left[\left[T(v_1) \right]_{B'}, \left[T(v_2) \right]_{B'}, \mathrm{L}\ , \left[T(v_n) \right]_{B'} \right] \in M_{m \times n}$$

Transformation matrix for nonstandard bases

Let V and W be finite-dimensional vector spaces with basis B and B', respectively, where $B = \{v_1, v_2, \mathrm{L}\ , v_n\}$

If $T : V \to W$ is a L.T. such that

$$\left[T(v_1)\right]_{B'} = \begin{bmatrix} a_{11} \\ a_{21} \\ \mathrm{M} \\ a_{m1} \end{bmatrix}, \quad \left[T(v_2)\right]_{B'} = \begin{bmatrix} a_{12} \\ a_{22} \\ \mathrm{M} \\ a_{m2} \end{bmatrix}, \mathrm{L}\ , \quad \left[T(v_n)\right]_{B'} = \begin{bmatrix} a_{1n} \\ a_{2n} \\ \mathrm{M} \\ a_{mn} \end{bmatrix}$$

the $m \times n$ matrix whose i columns correspond to $\left[T(v_i)\right]_{B'}$ *is*

$$A = \left[T(v_1) \mid T(v_2) \mid \mathbf{L} \mid T(v_n)\right] = \begin{bmatrix} a_{11} & a_{12} & \mathbf{L} & a_{1n} \\ a_{21} & a_{22} & \mathbf{L} & a_{2n} \\ \mathbf{M} & \mathbf{M} & \mathbf{O} & \mathbf{M} \\ a_{m1} & a_{m2} & \mathbf{L} & a_{mn} \end{bmatrix}$$

such that $\left[T(\mathbf{v})\right]_{B'} = A[\mathbf{v}]_B$ for every $\mathbf{v}$ in V.

- Ex : (Finding a transformation matrix relative to nonstandard bases)

Let T: $R^2 \to R^2$ be a L.T. defined by

$$T(x_1, x_2) = (x_1 + x_2, 2x_1 - x_2)$$

Find the matrix of T relative to the basis

$B = \{(1, 2), (-1, 1)\}$ and $B' = \{(1, 0), (0, 1)\}$

Sol:

$$T(1, 2) = (3, 0) = 3(1, 0) + 0(0, 1)$$

$$T(-1, 1) = (0, -3) = 0(1, 0) - 3(0, 1)$$

$$[T(1, 2)]_{B'} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}, \quad [T(-1, 1)]_{B'} = \begin{bmatrix} 0 \\ -3 \end{bmatrix}$$

the transformation matrix T relative to **B and B'**

$$A = \begin{bmatrix} [T(1, 2)]_{B'} & [T(-1, 1)]_{B'} \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & -3 \end{bmatrix}$$

***Now use* the matrix A to find $T(\mathbf{v})$, where $\mathbf{v} = (2, 1)$**

$$\mathbf{v} = (2,1) = 1(1,2) - 1(-1,1) \qquad B = \{(1,2), (-1,1)\}$$

$$\Rightarrow [\mathbf{v}]_B = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\Rightarrow [T(\mathbf{v})]_{B'} = A[\mathbf{v}]_B = \begin{bmatrix} 3 & 0 \\ 0 & -3 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

$$\Rightarrow T(\mathbf{v}) = 3(1,0) + 3(0,1) = (3,3) \qquad B' = \{(1,0), (0,1)\}$$

- Check: $T(2,1) = (2+1, 2(2) - 1) = (3,3)$

- **Notes:**

(1) In the special case where $V = W$ and $B = B'$,

the matrix A is called the matrix of T relative to the basis B

(2) $T : V \to V$: the identity transformation

$B = \{\vec{v}_1, \vec{v}_2, \mathrm{L}\ , \vec{v}_n\}$: **a basis for V**

$\Rightarrow$ **the matrix of T** relative to the basis B

$$A = \left[\left[T(\vec{v}_1)\right]_B, \left[T(\vec{v}_2)\right]_B, \mathrm{L}\ , \left[T(\vec{v}_n)\right]_B\right] = \begin{bmatrix} 1 & 0 & \mathrm{L} & 0 \\ 0 & 1 & \mathrm{L} & 0 \\ \mathrm{M} & \mathrm{M} & \mathrm{O} & \mathrm{M} \\ 0 & 0 & \mathrm{L} & 1 \end{bmatrix} = I_n$$

4.4 Transition Matrices and Similarity

$T: V \to V$ (a L.T.)

$B = \{v_1, v_2, \ldots, v_n\}$ (a basis of V)

$B' = \{w_1, w_2, \ldots, w_n\}$ (a basis of V)

$A = \left[[T(v_1)]_B, [T(v_2)]_B, \ldots, [T(v_n)]_B \right]$ (matrix of T relative to B)

$A' = \left[[T(w_1)]_{B'}, [T(w_2)]_{B'}, \ldots, [T(w_n)]_{B'} \right]$ (matrix of T relative to B')

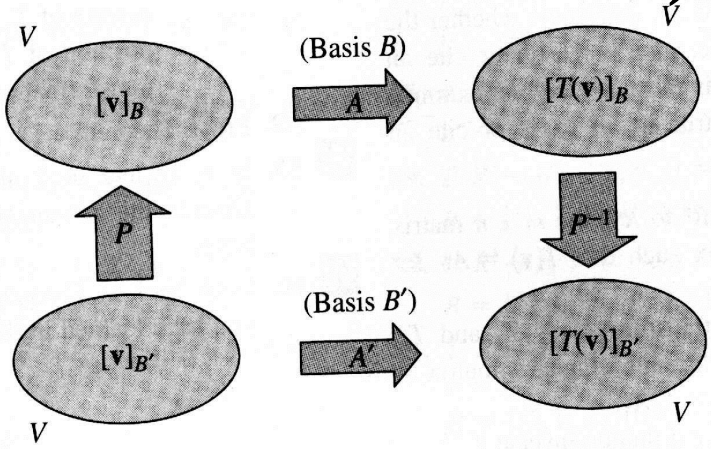
$P = \left[[w_1]_B, [w_2]_B, \ldots, [w_n]_B \right]$ (transition matrix from B' to B)

$P^{-1} = \left[[v_1]_{B'}, [v_2]_{B'}, \ldots, [v_n]_{B'} \right]$ (transition matrix from B to B')

$$\therefore [\mathbf{v}]_B = P[\mathbf{v}]_{B'}, \quad [\mathbf{v}]_{B'} = P^{-1}[\mathbf{v}]_B$$

$$[T(\mathbf{v})]_B = A[\mathbf{v}]_B$$

$$[T(\mathbf{v})]_{B'} = A'[\mathbf{v}]_{B'}$$



- **Two ways to get from $[\mathbf{v}]_{B'}$ to $[T(\mathbf{v})]_{B'}$:**

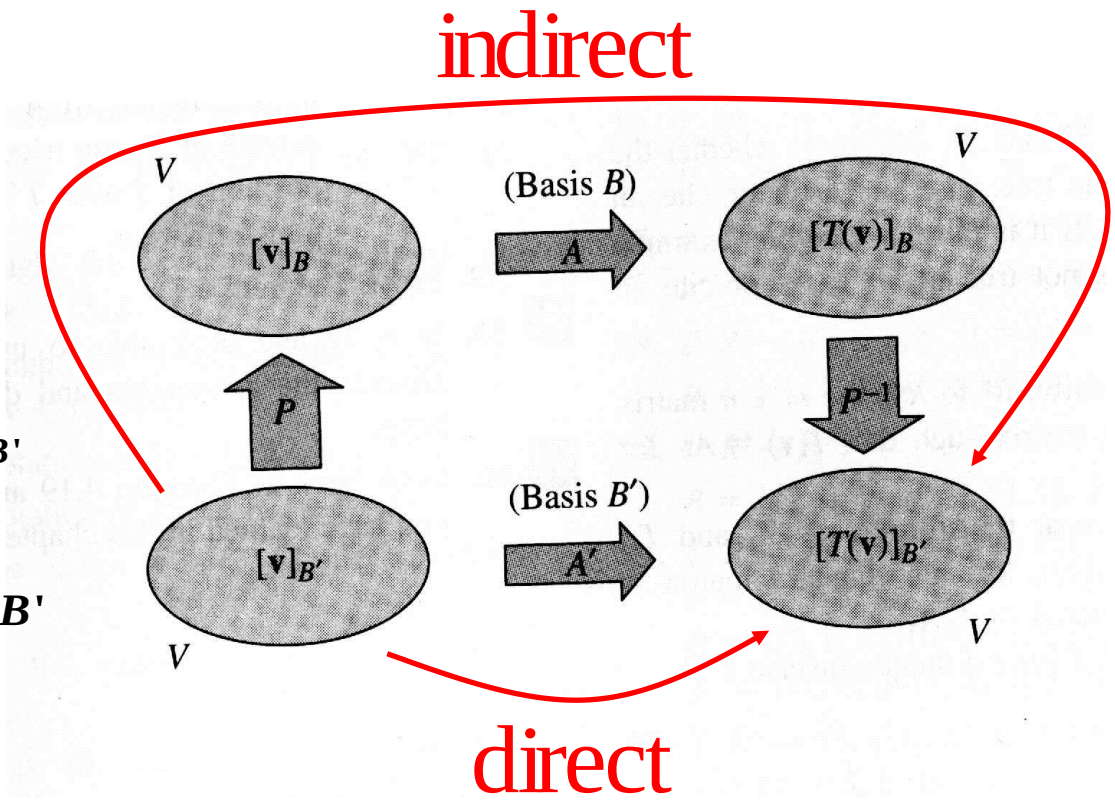
(1) direct

$$A'[\mathbf{v}]_{B'} = [T(\mathbf{v})]_{B'}$$

(2) indirect

$$P^{-1}AP[\mathbf{v}]_{B'} = [T(\mathbf{v})]_{B'}$$

$$\Rightarrow A'_{B'} = P^{-1}{}_{B'B} A_B P_{BB'}$$



- **Ex** Find the transformation matrix A' for $T: R^2 \to R^2$

$$\text{with } T(x_1, x_2) = (2x_1 - 2x_2,\ -x_1 + 3x_2)$$

reletive to the basis $B' = \{(1,\ 0),\ (1,\ 1)\}$

Sol:

$$\text{(I) } A' = \left[[T(1,\ 0)]_{B'} \quad [T(1,\ 1)]_{B'} \right]$$

$$T(1,\ 0) = (2,\ -1) = 3(1,\ 0) - 1(1,\ 1) \quad \Rightarrow \quad [T(1,\ 0)]_{B'} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

$$T(1,\ 1) = (0,\ 2) = -2(1,\ 0) + 2(1,\ 1) \quad \Rightarrow \quad [T(1,\ 1)]_{B'} = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$$

$$\Rightarrow A' = \left[[T(1,\ 0)]_{B'} \quad [T(1,\ 1)]_{B'} \right] = \begin{bmatrix} 3 & -2 \\ -1 & 2 \end{bmatrix}$$

(II) Standard matrix for T (*i.e.*, the transformation matrix of T relative to $B = \{(1, 0), (0, 1)\}$)

$$A = \begin{bmatrix} T(1,0) & T(0,1) \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ -1 & 3 \end{bmatrix} \quad \text{with} \quad T(x_1, x_2) = (2x_1 - 2x_2, -x_1 + 3x_2)$$

The transition matrix from B' to B: $P = \begin{bmatrix} [(1, 0)]_B & [(1, 1)]_B \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

The transition matrix from B to B': $P^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$

The transformation matrix of T relative $B'\{(1,0),(1,1)\}$

$$A' = P^{-1}AP = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & -2 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ -1 & 2 \end{bmatrix}$$

- **Similar matrix:**

For square matrices A and A' of order n, A' is said to be similar to A if there exist **an invertible matrix P** *such that* $A' = P^{-1}AP$

- **Thm 4.12:** (Properties of similar matrices)

Let A, B, and C be square matrices of order n.

Then the following properties are true.

(1) A is similar to A.

(2) If A is similar to B, then B is similar to A.

(3) If A is similar to B and B is similar to C, then A is similar to C.

Pf: $(1) A = I_n A I_n$

$$(2) A = P^{-1}BP \Rightarrow PAP^{-1} = P(P^{-1}BP)P^{-1}$$

$$PAP^{-1} = B \Rightarrow Q^{-1}AQ = B \ (Q = P^{-1})$$

- Ex : (A comparison of two matrices for a linear transformation)

Suppose $A=\begin{bmatrix}1 & 3 & 0\\ 3 & 1 & 0\\ 0 & 0 & -2\end{bmatrix}$ is the matrix for $T: R^3 \to R^3$ relative to the standard basis B. Find the matrix for T relative to the basis

$$B'=\{(1,1,0),(1,-1,0),(0,0,1)\}$$

Sol:

The transition matrix P from B' to the standard basis B is

$$P=\begin{bmatrix}[(1,\ 1,\ 0)]_B & [(1,\ -1,\ 0)]_B & [(0,\ 0,\ 1)]_B\end{bmatrix}=\begin{bmatrix}1 & 1 & 0\\ 1 & -1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$\Rightarrow P^{-1}=\begin{bmatrix}\frac{1}{2} & \frac{1}{2} & 0\\ \frac{1}{2} & -\frac{1}{2} & 0\\ 0 & 0 & 1\end{bmatrix}$$

The matrix of T relative to B' :

$$A' = P^{-1}AP = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}_{B \to B'} \begin{bmatrix} 1 & 3 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}_{B} \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{B' \to B}$$

$$= \begin{bmatrix} 4 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix} = \textit{diagonal matrix}$$